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The Pascal Hexagram.

By Miss Christine Ladd, *Johns Hopkins University.*

I wish to propose a new notation for the lines and points connected with the Pascal Hexagram, to give a brief account of the discoveries of Veronese on the subject and to develop a few additional properties of the figure.

I.

The vertices of the hexagon inscribed in the conic, S, are A, B, C, D, E, F; the lines tangent to the conic at these vertices respectively are a, b, c, d, e, f. In general, a large letter will represent a point, a small letter a line. Lines joining vertices of the inscribed hexagon are called fundamental lines; intersections of sides of the circumscribed hexagon are called fundamental points. The intersection of the two fundamental lines AB, DE is called $P(AB.DE)$; the line joining two fundamental points, ab, de, is called $p'(ab.de)$. It is evident that $p'(ab.de)$ is the pole of $P(AB.DE)$. There are 45 points P and 45 lines p'. The Pascal line obtained by taking the vertices of the hexagon in the order $ABCDEF$ is called $h(ABCDEF)$. It passes through the points $P(AB.DE)$, $P(BC.EF)$, $P(CD.FA)$. Similarly, the intersection of the lines $p'(ab.de)$, $p'(bc.ef)$, $p'(cd.fa)$ is the Brianchon point $H'(abcdef)$ of the hexagon $abcdef$, the pole of $h(ABCDEF)$.

The three Pascal lines which meet in a Steiner point are (Salmon's *Conic Sections*, 5th ed., note, p. 361) $h(ABCFED)$, $h(AFCDEB)$, $h(ADCBEF)$. We shall call the Steiner point in which they meet $G(ACE.BFD)$. In this symbol, the relative cyclic order of the letters in each group of three is all that it is necessary to observe; for instance, $G(AEC.FBD)$ and $G(ACE.BFD)$ are the same as $G(ACE.BFD)$. Given a G point, the h lines through it are obtained by taking one group of three in a fixed order for the odd letters and permuting cyclically the other group of three for the even letters. The Pascals which pass through the conjugate G point are $h(ABCDEF)$, $h(ADCFEB)$, $h(AFCBED)$, and the symbol of that G point is $G(ACE.BDF)$; hence two

G points whose notation differs in the particular that one group of three letters has suffered a change not cyclic are conjugate with respect to the conic S. There are ten ways in which six letters can be divided into two groups of three each, hence there are ten pairs of points G.

Four G points which lie on one Steiner-Plücker line are (Salmon, p. 362,) $G\,(BDA\,.\,ECF)$, $G\,(EDF\,.\,BCA)$, $G\,(BCF\,.\,EDA)$, $G\,(BDF\,.\,ECA)$. We shall call the Steiner-Plücker line on which they lie $i\,(BE\,.\,CD\,.\,AF)$. In the notation of an i line the division into groups of two is important, but not the order of the letters in each group. The number of ways in which six things can be separated into three different groups of two things each is fifteen, hence there are fifteen lines i. The G points on one i line are obtained by selecting one letter out of each group of two for the first group of three, and taking the remaining three letters, in the same order, for the other group of three. As this can be done in four different ways, there are four points G on one line i. Through one point G pass three lines i; viz., through $G\,(ABC\,.\,DEF)$, pass $i\,(AD\,.\,BE\,.\,CF)$, $i\,(AE\,.\,BF\,.\,CD)$, $i\,(AF\,.\,BD\,.\,CE)$. In writing the symbols for the i lines through one G point, it is necessary to observe that the cyclic order of the first letters of the three duads must be the same as that of the second letters; for instance, through $G\,(ABC\,.\,DEF)$ does not pass $i\,(AD\,.\,BF\,.\,CE)$.

The Kirkman point which corresponds to the Pascal $h\,(ABCDEF)$ is (Salmon, p. 363,) the intersection of the Pascal lines $h\,(ACEBFD)$, $h\,(CEADBF)$, $h\,(EACFDB)$. We shall call this the Kirkman point $H\,(ABCDEF)$. The Pascal lines through a Kirkman point are obtained by taking the three odd letters in the order in which they stand, and then the three even letters, inverting the order of the last two, for the first Pascal; and then deriving the other two Pascals from this by a cyclic change of the first three letters in one direction and of the last three in the other direction. Similarly, the three Kirkman points on one Pascal, $h\,(ACEBFD)$, are $H\,(AEFCDB)$, $H\,(EFABCD)$, $H\,(FAEDBC)$. If we wish to know whether two given Pascals, as $h\,(ABCDEF)$, $h\,(AEDBCF)$, intersect in a Kirkman point or not, we have to see if the same three letters stand together in each, in two groups which have suffered opposite cyclic changes. The two lines just written are $h\,(BCDEFA)$, $h\,(DBCFAE)$, and they meet in $H\,(BECADF)$.

The three H points of one Cayley-Salmon line are (Salmon, p. 362,) $H\,(ABCFED)$, $H\,(ADCBEF)$, $H\,(AFCDEB)$. We shall call this Cayley-

Salmon line, the line $g\ (ACE\,.\,BFD)$. It passes through the point $G\ (ACE\,.\,BDF)$. Through the two conjugate points $G\ (ACE\,.\,BDF)$, $G\ (ACE\,.\,BFD)$, pass respectively the lines $g\ (ACE\,.\,BFD)$, $g\ (ACE\,.\,BDF)$. These two g lines we shall call, for the present, corresponding g lines. They are not conjugate with respect to the conic S. (Veronese, *Nuovi Teoremi sul Hexagrammum Mysticum*, p. 26.) The H points on $G\ (ACE\,.\,BDF)$ correspond to the h lines through $g\ (ACE\,.\,BDF)$; hence we shall say that $g\ (ACE\,.\,BDF)$ corresponds to $G\ (ACE\,.\,BDF)$, while it passes through $G\ (ACE\,.\,BFD)$.

The symbol for the Salmon point in which four g lines intersect is obtained in the same way as that of the Steiner-Plücker line through four G points. In fact, the lines $g\ (BDA\,.\,ECF)$, $g\ (EDF\,.\,BCA)$, $g\ (BCF\,.\,EDA)$, $g\ (BDF\,.\,ECA)$, intersect in the Salmon point $I\ (BE\,.\,CD\,.\,AF)$; and the I points on $g\ (ACE\,.\,BDF)$, are $I\ (AB\,.\,CD\,.\,EF)$, $I\ (AD\,.\,CF\,.\,EB)$, $I\ (AF\,.\,CB\,.\,ED)$.

Professor Cayley (Quarterly Journal, Vol. IX,) gives a table to show in what kind of a point each Pascal line meets every one of the 59 other Pascal lines. By attending to the notation of Pascal lines such a table may be dispensed with. His 90 points "m," 360 points "r," 360 points "t," 360 points "z," and 90 points "w" are the intersections each of two Pascals whose symbols can easily be derived one from another. For instance,

$$\left.\begin{matrix} h\ (ABCDEF) \\ h\ (ABCFED) \end{matrix}\right> \text{“}m\text{,”} \qquad \left.\begin{matrix} h\ (DEFABC) \\ h\ (DEFBCA) \end{matrix}\right> \text{“}r\text{,”} \qquad \left.\begin{matrix} h\ (ACEDBF) \\ h\ (ABCDEF) \end{matrix}\right> \text{“}t\text{,”}$$

$$\left.\begin{matrix} h\ (ACFBED) \\ h\ (AFEDCB) \end{matrix}\right> \text{“}z\text{,”} \qquad \left.\begin{matrix} h\ (ACEBFD) \\ h\ (ABCDEF) \end{matrix}\right> \text{“}w\text{.”}$$

By producing the lines and points of the Brianchon hexagon, as we may call the corresponding circumscribed hexagon, we should find occasion for the same symbols, in small letters, for the H', G', I' points, which are the poles of the h, g, i lines, and for the h', g', i' lines, which are the poles of the H, G, I points.

It was shown by Kirkman that the two Kirkman points

$$H\ (BFCEAD), \quad H\ (BFDEAC),$$

are on a line through the point $P\ (AB\,.\,FE)$. I shall call this line $v_{12}\ (BF\,.\,EA)$ (and it happens that my notation here coïncides with that of Veronese, p. 43). So the points

$$H\ (BFCAED), \quad H\ (BFDAEC),$$

are on the line $v_{12}\ (BF\,.\,AE)$, which passes through $P\ (EB\,.\,FA)$ and which does not coïncide with $v_{12}\ (BF\,.\,EA)$. Through each point P pass two v_{12} lines,

viz., through $P(EB.FA)$ pass $v_{12}(EF.AB)$ and $v_{12}(EA.FB)$. There is but one P point on each v_{12} line. Through each H point pass three v_{12} lines; through $H(BCDEFA)$ pass $v_{12}(BC.EF)$, $v_{12}(CD.FA)$, $v_{12}(DE.AB)$. There are therefore $\frac{3.60}{2}$ or 90 v_{12} lines in all. If we look for the corresponding property of h lines, we find that

$$h(BFCEAD), \quad h(BFDEAC),$$

intersect in $P(BF.EA)$, and that

$$h(BFCAED), \quad h(BFDAEC),$$

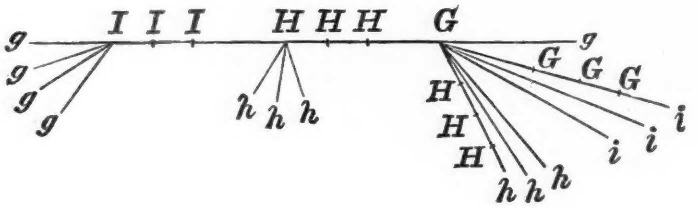
intersect in $P(BF.AE)$, but that $P(BF.AE)$ is the same point as $P(BF.EA)$. This is the intrinsic difference between H points and h lines. The H points lie in *twos* on 90 lines v_{12} which pass by threes through the 60 H points. The h lines intersect in *fours* in 45 points P, which lie in threes on the 60 h lines. To a P point, $P(BF.AE)$, may be said to correspond the pair of v_{12} lines, $v_{12}(BF.AE)$, $v_{12}(BF.EA)$. In the Brianchon hexagon, on the other hand, the H' points lie in fours on the 45 p' lines, and the h' lines intersect in twos in 90 points V'_{12}, which lie in threes on h' lines and in twos on p' lines. Not even in a hexagon which can be inscribed in one conic and circumscribed about another is there entire correspondence between Kirkman points and Pascal lines.

To resume:

To 60 Pascal	lines h	correspond	60 Kirkman	points	H.
" 20 Cayley-Salmon	" g	"	20 Steiner	"	G.
" 15 Steiner-Plücker	" i	"	15 Salmon	"	I.

On each h line lie three H's and one G.
" " g " lie three H's, three I's and one G.
" " i " lie four G's.
Through each H point pass three h's and one g.
" " G " pass three h's, three i's and one g.
" " I " pass four g's.

The whole arrangement can be diagrammatically represented by a simple figure:



On $h\,(ABCDEF)$ lie $H\,(ACEBFD)$, $H\,(CEADBF)$, $H\,(EACFDB)$;
and $G\,(ACE.BDF)$.

" $g\,(ABC.DEF)$ " $H\,(ADBECF)$, $H\,(AFBDCE)$, $H\,(AEBFCD)$;
$I\,(AD.BE.CF)$, $I\,(AE.BF.CD)$, $I\,(AF.BD.CE)$;
and $G\,(ABC.DFE)$.

" $i\,(AB.CD.EF)$ " $G\,(ACE.BDF)$, $G\,(ACF.BDE)$, $G\,(ADE.BCF)$,
$G\,(ADF.BCE)$.

Through $H\,(ABCDEF)$ pass $h\,(ACEBFD)$, $h\,(CEABDF)$, $h\,(EACFDB)$;
and $g\,(ACE.BDF)$.

" $G\,(ABC.DEF)$ " $h\,(ABDECF)$, $h\,(AFBDCE)$, $h\,(AEBFCD)$;
$i\,(AD.BE.CF)$, $i\,(AE.BF.CD)$, $i\,(AF.BD.CE)$;
and $g\,(ABC.DFE)$.

" $I(AB.CD.EF)$ " $g\,(ACE.BDF)$, $g\,(ACF.BDE)$, $g\,(ADE.BCF)$,
$g\,(ADF.BCE)$.

II.

By the notation here given it is immediately evident what points are on every line and what lines pass through every point, without referring to tables, as Veronese is obliged to do. I shall make use of this notation, so far as any notation is necessary, in describing Veronese's additions to the subject.

Pascal discovered the theorem which bears his name in 1640. The reciprocal theorem of Brianchon remained unknown until 1806. From the time, 1828, when Steiner showed that by taking the six points on the conic in different orders, sixty Pascal lines may be obtained, the development of the figure has been more rapid. Steiner himself showed that the 60 Pascal lines meet in threes in the 20 Steiner points, and he believed that these points were situated in fours on five lines meeting in one point. Plücker showed that they lie in reality on fifteen lines, three through each point. Hesse observed that the 20 Steiner points consist of ten pairs of points harmonically conjugate with respect to the conic, and that the figure of the Steiner points and the Steiner-Plücker lines is identical with that formed by three triangles in perspective. Kirkman showed that the Pascal lines pass by threes through the sixty points called by his name, and that these points are

connected two and two by 90 lines v, which pass each through two points P. Professor Cayley and Dr. Salmon discovered at the same time that the 60 Kirkman points lie in threes on 20 (Cayley-Salmon) lines g, and Dr. Salmon, that the lines g meet in threes in the 15 (Salmon) points I. Hesse pointed out the correspondence which exists between the lines and points of the figure; but he was aware that the relation is not that of pole and polar, at least not with respect to the original conic. (Crelle, Vol. 68, p. 193.)

Veronese has written a paper (*Nuovi Teoremi sul Hexagrammum Mysticum*, Reale Accademia dei Lincei, 1876–1877,) which apparently leaves little work for other investigators to do. His most important discovery is that the 60 h lines may be divided into six groups of ten lines each, which intersect in the ten corresponding H points and are their polars with respect to a conic π. There are six conics π in the whole figure, and any five of these groups of ten lines and points determine the sixth. He has shown, moreover, that besides the original system, $[H_1h_1]$, of 60 Pascal lines and Kirkman points, there is an infinity of such systems, $[H_nh_n]$, consisting each of six groups of ten lines and points, and giving rise each to six conics. Five groups of any system after the first suffice to determine one group of the preceding and one of the succeeding system. The figure of the g lines and of the G points is common to all these systems; that is to say, the 60 H points of every system lie in threes on the same 20 g lines and the 60 h lines of every system pass by threes through the same 20 G points. It follows that the I points and the i lines are also common to all the systems. Veronese uses the symbol π for a group of ten lines and points as well as for the conic with respect to which they are poles and polars. He gives a table by consulting which one can see to what figure π any h line belongs. But the h lines which go together to form a figure π can be determined at once by observing the following rule: Take any h line, the other six h lines through the three H points on it, and the three h lines through the H point which corresponds to it; these ten h lines constitute a figure π, to which belong also the ten H points of the same notation. A symbol for a figure π thus obtained, from which symbol it can be known immediately whether a given line or point belongs to the figure which it represents or not, is a desideratum. Veronese calls his figures π first, second, third, &c., and the connection between the first figure and its lines and points is of course entirely arbitrary. No two h lines of one figure π pass through a common G point, hence to a figure π correspond ten differ-

ent G points. No two of them are conjugate G points. Any two figures π have in common four G points which lie on one i line, or to each i line corresponds one of the 15 possible combinations two by two of the six figures π; and the four g lines common to any two figures π pass through an I point.

The connecting link between the system $[H_1h_1]$, and the system $[H_2h_2]$, is formed by the 90 lines v_{12}, which fact is indicated by the suffix $_{12}$. We have already seen that the v_{12} lines which pass through $H_1\,(BCDEFA)$ are

$$v_{12}\,(BC.EF),\quad v_{12}\,(CD.FA),\quad v_{12}\,(DE.AB).$$

Now three v_{12} lines which pass through one H_2 point are (Veronese, p. 35)

$$v_{12}\,(BC.FE),\quad v_{12}\,(CD.AF),\quad v_{12}\,(DE.BA).$$

That is, given three pairs of v_{12} lines such that one member of each pair passes through a common H_1 point, the remaining members pass through a common H_2 point. This correspondence between H_1 points and H_2 points I shall indicate by giving two such points the same notation. It will then be observed that the three v_{12} lines of one H_1 point are obtained by taking its opposite pairs of letters in the order in which they stand; but the three v_{12} lines of one H_2 point by taking opposite pairs of letters with an inversion of one pair. On a v_{12} line, $v_{12}\,(AB.CD)$, lie two H_1 points, $H_1\,(ABECDF)$, $H_1\,(ABFCDE)$, and two H_2 points, $H_2\,(ABEDCF)$, $H_2\,(ABFDCE)$.

The three H_2 points which have the same notation as the three h_1 lines of an H_1 point lie on an h_2 line (Veronese, p. 39). Through each H_2 point pass three h_2 lines. There are 60 H_2 points and 60 h_2 lines.

Two lines h_2 of the same notation as the two H_1 points of one v_{12} line meet in a point V_{23}, through which pass two h_3 lines of the third system $[H_3h_3]$. These h_3 lines, 60 in number, determine by their intersections in threes the 60 H_3 points, which lie in threes on the h_3 lines. There are 45 *pairs* of points V_{23}, answering to the 45 points P of the system $[H_1h_1]$; that is to say, after the first system the intrinsic difference between H points and h lines drops out, or h lines no longer meet by fours in 45 points, but by twos in 90 points.

In general, from the system $[H_{2n-1}h_{2n-1}]$ the system $[H_{2n},\,h_{2n}]$ is derived by means of lines $v_{2n-1,\,2n}$, the connectors of pairs of H_{2n-1} points and also of pairs of H_{2n} points. From the system $[H_{2n}h_{2n}]$ we pass to the system $[H_{2n+1}h_{2n+1}]$ by means of points $V_{2n,\,2n+1}$, the intersections of pairs of h_{2n} lines and also of pairs of h_{2n+1} lines.

All the pairs of v lines of same notation but from different systems, $v_{12}\,(AB.CD)$, $v_{12}\,(AB.DC)$; $v_{34}\,(AB.CD)$, $v_{34}\,(AB.DC)$; $v_{56}\,(AB.CD)$,

v_{56} $(AB.DC)$, &c., meet in a single point Y, through which passes also an i line; and all the pairs of points of the same notation, V_{23}, V_{45}, V_{67}, &c., together with the P point of the same notation, lie in a line y, through an I point. There are 45 lines y, three through each I point, and 45 points Y, three on each i line (p. 52).

The 90 points V_{23}, V_{45}, &c., also lie in twos on 180 lines n_{23}, n_{45}, &c., respectively, which pass by fours through the 45 points Y. A similar relation holds between the v lines (p. 60).

Veronese gives many relations of harmonicism and of involution, which I omit. For instance, he shows that the pairs of points H_2H_3, H_4H_5, H_6H_7, &c., of same notation, which lie all on a common g line, form a system of points in involution, whose double points are the H point of same notation and the I point of the g line.

III.

1. Since the point $G\,(ABC.DEF)$ is conjugate to the point $G\,(ABC.DFE)$ with respect to the conic S, and the pole of the line g' $(abc.def)$ with respect to the same conic, it follows that the point G $(ABC.DFE)$ is on the line g' $(abc.def)$; it is also on the line g $(ABC.DEF)$, hence it is at their intersection. In general, g lines and g' lines of the same notation intersect in G points. Since in the Brianchon figure the g' lines consist of ten pairs of lines conjugate with respect to S, it may be shown in the same way that G points and G' points of the same notation, as G $(AFC.BED)$ and G' $(afc.bed)$, lie on g' lines, as g' $(afc.bde)$.

2. Since $ABCD$ is a quadrilateral inscribed in a conic, the intersections of its diagonals, P $(BC.AD)$, P $(CD.AB)$, P $(AC.BD)$, are the vertices of a triangle self-conjugate to the conic and the line joining P $(CD.AB)$ to P $(AC.BD)$ is the polar of P $(BC.AD)$; but p' $(bc.ad)$ is also the polar of P $(BC.AD)$, hence these two lines coïncide. In the same way it may be shown that the point of intersection of p' $(cd.ab)$ and p' $(ac.bd)$ coïncides with P $(BC.AD)$, and, in general, that the triangle whose vertices are the P points obtained from four of the six points on the conic coïncides with the triangle whose sides are the p' lines obtained from the tangents at the same four points. There are 15 combinations of four letters out of six, hence there are 15 of these self-conjugate triangles. Since a self-conjugate triangle has

always one vertex within the conic and two without, it follows that 15 P points are always within the conic and 30 without, and that of the 45 p' lines 30 cut the conic in two real and 15 in two imaginary points.

It now appears that the lines and points of the Brianchon figure can be produced without considering the Brianchon hexagon. Since the points $P(AB.DE)$, $P(BC.EF)$, $P(CD.FA)$ are on a line, $h(ABCDEF)$, their poles, $P(AD.EB)$ $P(AE.DB)$, $P(BE.FC)$ $P(BF.CE)$, $P(CF.AD)$ $P(CA.DF)$, meet in a point [the same as the point $H'(abcdef)$], the pole of $h(ABCDEF)$. From the 60 points thus obtained may be produced all the other lines and points of the figure.

3. If a, b, c be the sides of a triangle and a', b', c', d' the sides of a quadrilateral such that the triangles $b'c'd'$, $c'd'a'$, $d'a'f'$, $a'b'c'$ are homologous with abc, their respective axes of homology being k_a, k_b, k_c, k_d, then the intersections of k_a, a'; k_b, b'; k_c, c'; k_d, d' are collinear. For, the equations of the axes may be written $k_a)$ $b + c' = c + b' = a + d' = 0$, $k_b)$ $b + d' = c + a' = a + c' = 0$, $k_c)$ $b + a' = c + d' = a + b' = 0$, $k_d)$ $b + b' = c + c' = a + a' = 0$, and we shall then have for lines through their respective intersections with sides of the quadrilateral

$$
\begin{aligned}
k_a,\ a')\ & b + c' + a' = c + b' + a' = a + d' + a' = 0,\\
k_b,\ b')\ & b + d' + b' = c + a' + b' = a + c' + b' = 0,\\
k_c,\ c')\ & b + a' + c' = c + d' + c' = a + b' + c' = 0,\\
k_d,\ d')\ & b + b' + d' = c + c' + d' = a + a' + d' = 0,
\end{aligned}
$$

which form all four one and the same line. The quadrilateral $k_a k_b k_c k_d$ is also such that its four triangles are each homologous with abc, and in fact in such a way that $k_a k_b k_c$, $a'b'c'$ and abc have lines joining all three corresponding vertices coïncident. Take the triangles $k_b k_c k_d$ and $b'c'd'$; we have

$$k_c - k_b = b - c = b' - c' = 0, \qquad -k_c - k_d = -a - b = -d' + c' = 0,$$

$k_b + k_d = a + c = d' - b' = 0$, and the equations show that these three lines meet in a point. Let us apply this property to the Pascal hexagram. We shall say, with Veronese, (p. 27), that the triangle formed by joining opposite vertices of a hexagon *belongs* to the Pascal obtained by taking the vertices in the same order; for instance, the triangle whose sides are AD, BE, CF, belongs to the four Pascals $h(AECDBF)$, $h(AEFDBC)$, $h(ABFDEC)$, $h(ABCDEF)$. The points

$P(AB.DE)$,	$P(AD.BE)$	are on	the line	$p'(bd.ae)$;
$P(BC.EF)$,	$P(EB.FC)$	"	"	$p'(ce.bf)$;
$P(FD.AC)$,	$P(AD.FC)$	"	"	$p'(af.dc)$.

These three p' lines meet in a point, namely, the Brianchon point $H'\,(aecdbf)$, hence the two triangles formed by the vertical rows of P points are homologous. The sides of the first are the Pascals $h\,(ABFDEC)$, $h\,(ABCDEF)$, $h\,(CAEFDB)$, and the corresponding sides of the second are AD, BE, CF, hence these three pairs of sides intersect in a line which, as it is an axis of homology corresponding to the centre of homology $H'\,(aecdbf)$, we shall call the line $k\,(AECDBF)$. In the same way it may be shown that the triangle formed by any three of the four Pascals to which the triangle AD, BE, CF belongs are homologous therewith, therefore the intersections of the four axes of homology, $k\,(AECDBF)$, $k\,(AEFDBC)$, $k\,(ABFDEC)$, $k\,(ABCDEF)$ with the four Pascal lines $h\,(AECDBF)$, $h\,(AEFDBC)$, $h\,(ABFDEC)$, $h\,(ABCDEF)$ respectively, are four points on one straight line. As this line is obtained by means of the triangle AD, BE, CF, we shall call it the line $l\,(AD\,.\,BE\,.\,CF)$. To each triangle formed by three fundamental lines, no two of which pass through the same point on the conic, corresponds a line l of the same notation; there are 15 such triangles, hence there are 15 lines l. To each H' point corresponds a k line, hence there are 60 lines k, divided into 15 groups of four each, which intersect corresponding h lines on the 15 l lines.

4. The triangles ABC, abc, are homologous. Let us call their centre of homology $C\,(ABC\,.\,abc)$, their axis $a\,(ABC\,.\,abc)$. Let us say that the points $C\,(ABC\,.\,abc)$, $C\,(ADC\,.\,adc)$ are joined by the line $c\,(\overline{\mathrm{ac}}\,.\,\mathrm{bd})$ and that the lines $a\,(ABC\,.\,abc)$, $a\,(ADC\,.\,adc)$ intersect in $A\,(\overline{\mathrm{ac}}\,.\,\mathrm{bd})$, where the bar is drawn over the letters that are repeated. I have shown (*Educational Times*, Question 5698,) that $c\,(\overline{\mathrm{ac}}\,.\,\mathrm{bd})$, $c\,(\mathrm{ac}\,.\,\overline{\mathrm{bd}})$ intersect in $P\,(AC\,.\,BD)$, and that $A\,(\overline{\mathrm{ac}}\,.\,\mathrm{bd})$, $A\,(\mathrm{ac}\,.\,\overline{\mathrm{bd}})$ are connected by $p'\,(ac\,.\,bd)$. There are 20 points C and 20 lines a. Each C point is joined to 9 other C points by c lines, hence there are $\frac{1}{2}\,(9\,.\,20) = 90$ lines c, which pass by twos through the 45 points P, and 90 points A which lie in twos on the 45 lines p'. The six c lines

$$\begin{matrix} c\,(\overline{\mathrm{ac}}\,.\,\mathrm{bd}) \\ c\,(\mathrm{ac}\,.\,\overline{\mathrm{bd}}) \end{matrix}, \qquad \begin{matrix} c\,(\overline{\mathrm{eb}}\,.\,\mathrm{fa}) \\ c\,(\mathrm{eb}\,.\,\overline{\mathrm{fa}}) \end{matrix}, \qquad \begin{matrix} c\,(\overline{\mathrm{df}}\,.\,\mathrm{ce}) \\ c\,(\mathrm{df}\,.\,\overline{\mathrm{ce}}) \end{matrix}$$

intersect in pairs in three points on one straight line, viz., the P points on $h\,(ACEBDF)$, hence they form the sides of a Pascal hexagon; and for a similar reason the six A points of the same notation are the vertices of a Brianchon hexagon.

5. The Brianchon hexagon formed by joining alternate vertices of $ABCDEF$ has for its sides AC, BD, CE, DF, EA, FB. The conic inscribed

in this hexagon, Σ_1, is the reciprocal of the conic S with respect to a third conic, X_1, twelve points of which may be obtained by taking on each side of the Brianchon hexagon the two points which form a harmonic range with each of the two pairs of vertices on this side; for instance, on AC the two points which are harmonic at once with C, $P(BD.AC)$, and with A, $P(BF.AC)$. The hexagon $ABCDEF$ is the reciprocal with respect to the conic X of the hexagon formed by joining its alternate vertices; the point $P(BD.AC)$ is the pole of the line BC, the point $P(AE.FD)$ is the pole of of the line FE, hence the Pascal $h(CAEBDF)$ is the polar of the point $P(BC.EF)$; $P(BD,CE)$ is the pole of CD, $P(FB.AE)$ is the pole of AF, hence the Pascal $h(AECFBD)$ is the polar of the point $P(CD.AF)$. It follows that the intersection of the Pascals $h(CAEBDF)$, $h(AECFDB)$, which is the Kirkman $H(AFEDCB)$, is the pole of the line joining $P(CD.AF)$ to $P(BC.FE)$, which is the Pascal $h(AFEDCB)$. But the six hexagons, $ABCDEF$, $AFCBED$, $ADCFEB$, $ABCFED$, $ADCBEF$, $AFCDEB$, form, by connectors of alternate vertices, a Brianchon hexagon composed of the same sides in different orders, and hence circumscribed to the same conic, therefore the six Pascals $h(ABCDEF)$, $h(AFCBED)$, $h(ADCFEB)$, $h(ABCFED)$, $h(ABDCEF)$, $h(AFCDEB)$, are the poles of the six Kirkmans $H(ABCDEF)$, $H(AFCBED)$, $H(ADCFEB)$, $H(ABCFED)$, $H(ADCBEF)$, $H(AFCDEB)$, with respect to the same conic X_1. Moreover, the points $G(ACE.BDF)$ and $G(ACE.BFD)$ in which the first three and the second three Pascals intersect are the poles respectively of the lines $g(ACE.BDF)$ and $g(ACE.BFD)$ which connect the first three and the second three Kirkmans. The two G points in question are harmonic conjugates with respect to the conic S, hence their polars with respect to X_1, the g lines of the same notation, are harmonic conjugates with respect to the reciprocal conic, Σ_1. The triangle whose vertices are two corresponding G points and the intersection of the g lines through them (or, what is the same thing, the triangle whose sides are two corresponding g lines and the line joining the G points on them) is a triangle self-conjugate with respect to the conic X_1, two of its vertices being at the same time conjugate with respect to S, and two of its sides with respect to Σ_1. Since this conic, Σ_1, is inscribed in the triangles ACE and BDF, we shall call it the conic $\Sigma(ACE.BDF)$, (where the order of the letters in each group of three is of no consequence) and the conic with respect to which it is the reciprocal of S we shall call $X(ACE.BDF)$. There are ten conics Σ, the

reciprocals of S with respect to ten conics X. Each pair of corresponding G points and the six Pascals through them are reciprocal, with respect to one conic X, to the two g lines and the six H points of the same notation. The 60 H points and the 60 h lines are then divided into ten systems of six lines and points each, reciprocal to each other with respect to the X conic of that system.

These properties of the Pascal Hexagram can be summed up in the following propositions:

(*1*). *The 20 Steiner points G are the intersections of the 20 Cayley-Salmon lines g with the 20 corresponding lines g'. The 20 lines g' are the connectors of the 20 Steiner points G with the 20 corresponding points G'.*

(*2*). *The 45 points P lie in twos on 45 lines p', which meet by threes in 60 points H', the poles with respect to the original conic of the h lines. The H' points lie in fours on the lines p', in threes on 60 lines h' and in threes on 20 lines g'. From them may be produced any number of systems of points and lines, $[H'_n h'_n]$ having their g' and i' lines and their G' and I' points in common. But in this case transition is made from a system of even index to one of odd by means of v' lines, and from one of odd to one of even by means of V' points.*

(*3*). *Three Pascal lines which belong to a triangle formed of fundamental sides intersect those sides in a k line. There are 60 lines k. Their intersections with corresponding h lines lie in fours on 15 lines l.*

(*4*). *Of the corresponding circumscribed and inscribed triangles of the conic, the 20 centres of homology, C, lie in twos on 90 lines c, which pass by twos through the 45 points P, and the 20 axes of homology, a, intersect in twos in 90 points A, which lie in twos on the 45 lines p'.*

(*5*). *The H points and the h lines may be divided into ten groups of six lines and points each. The lines and points of each group are poles and polars with respect to one of ten auxiliary conics X. To each group belong two corresponding G points and two corresponding g lines. They form a triangle self-conjugate with respect to the X conic of the group. The G points are at the same time conjugate with respect to conic S, and the g lines are at the same time conjugate with respect to the conic Σ, the X reciprocal of S.*

*On the Theory of Flexure.**

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It is not intended in this discussion to give the *exact* theory of flexure for all materials and shapes of pieces subjected to bending, nor indeed for any one kind of material. The present state of knowledge regarding the internal molecular action developed in any piece of elastic material by the action of external forces, is not such as to enable one to treat any problem of this kind with mathematical rigor if the piece be of finite dimensions. The illustrious Lamé, however, has remarked that the exact solutions of all problems in natural science are usually obtained by successive approximations, and such has certainly been the case in regard to the theory of flexure.

If the following investigation shall be found to constitute even a short step in the direction of the correct theory, the object of the writer will have been accomplished.

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* An explanation, by the writer, in regard to his aim in this discussion, is very essential in order that the results may not be misunderstood. It is not intended to cover any of the ground gone over so elegantly by St. Venant, Clebsch and others. Their investigations leave nothing to be desired.

It is intended to point out considerations which, it is believed, will account for the great discrepancies existing between the results of the "common theory" and those of experiment. Those considerations apply chiefly to the conditions of stress existing between the elastic limit and rupture, to which the investigations of the authors mentioned above do not apply.

It may easily be shown that the logarithmic law found is not consistent with the equations of condition (4), (5), (6) and (7) for a body of homogeneous elasticity, but those equations do not obtain beyond the elastic limit, nor for bodies that are not homogeneous (and non-homogenity is characteristic of all bodies used by the engineer), nor indeed are they strictly true for homogeneous bodies except for indefinitely small strains. Now indefinitely small strains are by no means those which accompany the application of finite external forces or the existence of finite internal stresses.

Again the researches of M. Tresca, in particular, but also those of Prof. Thurston and others * show that molecules rearrange themselves, to a greater or less extent, when the material in which they exist is subjected to stress for a finite length of time. It is not only possible, but highly probable, that this rearrangement enables the molecules to take such positions as will give the material the greatest possible capacity of resistence.

It is submitted, therefore, that, while it is altogether probable that that condition will exist just before rupture, which, by the principle of least resistance, will subject the material to the least stress, the same law, on the further investigation of strains in either homogeneous or non-homogeneous bodies, *may* be found to hold in the case of such bodies in equilibrium. For that reason some approximate values for the deflection are found which may serve the purpose of (at least) a rough experimental test.

The importance of the bearing of these matters on elastic bodies, is enhanced by the fact that no law of stress whatever can exist in such bodies in equilibrium which may not be supposed to exist in a rigid body.

The arbitrary functions of integration in u, v and w are not all found, for they are not needed for the purposes of the investigation, and a search for them would cause the paper to reach far beyond its proper limits.

[* As, for instance, Eaton Hodgkinson, who, we believe, made accurate determinations in this subject many years before those whose names are above mentioned, having turned his attention to it as early as 1824.—EDS.]

It is assumed, and assumed only in the "Common Theory of Flexure" put forth by Mariotte and Leibnitz, that the intensities of the normal internal stresses parallel to the neutral surface vary directly as the first powers of the normal distances from that neutral surface. This assumption gives results corresponding to experimental ones, with degrees of approximation varying according to the nature of the material and the shape of the piece subjected to bending. Its chief merit, and a very great one, is that it leads to very simple discussions of the cases which ordinarily occur in practice. It ignores, however, the existence of any internal shearing stress, and the formulae deduced for deflection do not involve the distortion which any piece of material suffers when subjected to the action of external forces.

Nevertheless, the method of fixing the position of the neutral surface is correct, since it is based on one of the first principles of statics, *i. e.*, that each of the sums of the components of the internal stresses, taken along three rectangular axes, must be equal to zero. The sum of the component forces of each sign along any axis, and not the sum of the component moments, must be equal to each other when the external forces act in a direction normal to the axis of the beam.

Navier first assumed the equality of the moments, but soon after abandoned the idea and pronounced it erroneous.

The principle just stated, first given by Parent, will be used in the following discussion in the determination of the position of the neutral surface.

Two assumptions will be made, the last only of which, however, as will eventually be shown, tends to give the investigation an approximate character.

The one source of approximation which probably causes the discrepancy between the results which follow and those of experiments is the neglect of lateral contraction and expansion; and those phenomena will be noticed further on.

It will first be assumed that the material has a non-crystalline structure. This is not absolutely necessary, but it emphasizes the proof that the results apply to material of any kind.

The second assumption is this, that the applied bending forces produce no compression at their points of application. This really amounts to supposing the bending to be produced by a single force acting at the proper distance from the section under consideration, while the portion of the beam on the other side of the section is held in position by the requisite forces.

If this assumption were a cause of approximation in the results, those results would not be essentially changed thereby in all ordinary cases of engineering practice as the compression is very slight.

In the case of glass the experiments of the late Mr. Louis Nickerson, C. E., of St. Louis, would seem to show that a high intensity of local pressure at the point where the external force is applied causes the neutral surface to move toward that point through an appreciable distance.

The general equations of equilibrium, however, do not indicate such a result, and there are strong reasons for believing that his experiments may have indicated something different.

The first assumption made renders it possible to make use of Lamé's general equations for homogeneous solids of constant elasticity. These are found on page 65 of his "Leçons sur le *théorie mathématique de l'elasticité des corps solides*," and are the following. Let u, v and w be the actual displacements of any molecule along any assumed three rectangular axes of x, y and z; then N_1, N_2 and N_3 represent the three normal intensities of stresses along these axes respectively, and T_1, T_2 and T_3 the intensities of tangential stresses producing moments around the same axes, *i. e.* T_1 around x, T_2 around y, and T_3 around z. Let λ and μ represent empirical constants depending on the nature of the material, and let $\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$. This quantity θ will be recognized as the dilitation per unit of volume. Using this notation, the general equation for a homogeneous solid are

$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu\frac{du}{dx}, \quad & T_1 &= \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ N_2 &= \lambda\theta + 2\mu\frac{dv}{dy}, \quad & T_2 &= \mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ N_3 &= \lambda\theta + 2\mu\frac{dw}{dz}, \quad & T_3 &= \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

No demonstration of these equations is given, for it is difficult to conceive of one more elegant or more general than that given by Lamé.

Neglecting the effect of forces emanating from an exterior centre, the conditions of equilibrium are involved in the following equations, also given by Lamé,

$$\left.\begin{aligned} \frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} &= 0 \\ \frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} &= 0 \\ \frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

These are the only equations of condition resulting from the consideration of the principles of statics alone, and are, in general, insufficient to determine the six unknown intensities which enter them.

In the following discussion the piece or beam subjected to bending will be supposed to occupy a horizontal position; the bending forces (including the reäction) will be supposed to act in a direction normal to the axis of the beam; the beam will be supposed straight and uniform in normal section; the axis of x will be taken to be parallel to the axis of the beam; the axis of z will be vertical and the axis of y horizontal and perpendicular to that of x. The axes of z and y will thus be parallel to axes of symmetry of the section, if that section be symmetrical and the beam be properly placed. No other kind of section or position will be considered. In the generality of cases the coefficients of elasticity for tension and compression will be considered equal. In the one or two cases where they are not supposed to be equal, the axis of x will still be taken parallel to the axis of the beam, and not coincident with it.

Now in the case of flexure, generally considered, on account of the distortion of the material subjected to stress, the six stresses N_1, N_2, N_3, T_1, T_2, T_3 actually exist, but in some of the cases taken some of them are equal to zero; in others, some of them are so small that they may be considered differential quantities, *i. e.*, they owe their existence to the indefinitely small difference of the intensities of stresses on two small portions of the material indefinitely close together. The omission of these quantities will evidently produce no essential error in the results, though it is true that it takes from the mathematical exactness of the equations.

Beams whose sections, *i. e.* normal sections, are symmetrical in respect to the axis of y and z will first be considered, and it will be assumed that $N_2 = 0$, $N_3 = 0$, and $T_1 = 0$. It should be stated that the sections considered will not only be symmetrical ones but such that they will not have re-entrant contours.

The case of rectangular sections when N_3 is not equal to zero will be taken up afterwards. It might be treated as existing in all beams if the external forces were so applied that T_1 is still zero, but that is an exceptional case and will not be taken up. T_1 may in reality exist as a very small quantity, in some cases, on account of the variable value of T_2 at the neutral surface.

The equations of condition for equilibrium in these cases, from equations 2, will be the three following:

$$\left.\begin{aligned} \frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} &= 0, \\ \frac{dT_3}{dx} &= 0, \\ \frac{dT_2}{dx} &= 0, \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

or

$$\left.\begin{aligned} (\lambda + \mu)\left(\frac{d^2u}{dx^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz}\right) + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) &= 0, \\ \mu\left(\frac{d^2u}{dxdy} + \frac{dv^2}{dx^2}\right) &= 0, \\ \mu\left(\frac{d^2w}{dx^2} + \frac{d^2u}{dxdz}\right) &= 0. \end{aligned}\right\} \quad . \ . \ . \quad (4)$$

Three other equations of condition result from the conditions that N_2, N_3 and T_1 each equal zero. These give in connection with equations (1)

$$\lambda\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + 2\mu\frac{dv}{dy} = 0, \quad . \ . \ . \ . \ . \ . \ . \quad (5)$$

$$\lambda\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + 2\mu\frac{dw}{dz} = 0, \quad . \ . \ . \ . \ . \ . \ . \quad (6)$$

$$\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) = 0. \quad . \ . \ . \ . \ . \ . \ . \quad (7)$$

These equations, as it will afterwards be seen, aid in the determination of the displacements u, v and w. The last two of equations (3) may be integrated at once, and will give

$$T_3 = f\,(y, z) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

$$T_2 = F\,(y, z) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

In which f and F signify any arbitrary functions of y and z whatever; they correspond to the "constants" of integration and must be written because the intensities of the internal stresses are, in general, each functions of x, y and z.

Denoting by $f_y'\,(y, z)$ and $F_z'\,(y, z)$ the partial derivatives of T_3 and T_2, respectively, in respect to the variables indicated, the first of equations (3) may be integrated, and will give

$$N_1 = -\,x\,[f_y'\,(y, z) + F_z'\,(y, z)] + \Psi\,(y, z) \quad . \ . \ . \ . \quad (10)$$

The quantity $\Psi\,(y, z)$ is any arbitrary function of y and z, and it will now be shown that in general it is independent of y and z, as well as of x, and that in many of the cases of pure flexure it may be put equal to zero.

The direction of action of the stress whose intensity is N_1 is normal to its plane of action, which is a normal section of a fibre parallel to the axis of the beam. Now, since the applied bending forces are perpendicular in direction to the axis of the beam, no part of N_1 can result directly from the forces; that is, they have no component parallel to the fibres subjected to the normal stress N_1.

The stress, whose intensity is N_1, exists *only*, therefore, in consequence of the shearing, or tangential, stresses called into action by the slipping over each other of the fibres parallel to the axis of the beam, or in consequence of T_2 and T_3. The expression for N_1 cannot therefore have a part independent of the quantities T_2 and T_3, except in the case (not of pure flexure) where the beam is subjected to the action of an external force acting in the direction of its own length. The function $\Psi(y, z)$ cannot, therefore, depend on the variables y and z unless they appear raised to the zero power; or, in other words, $\Psi(y, z)$ cannot exist except as a constant, since the integrating equation (10) was made in respect to x. But the case treated is that of pure flexure, in which no external force acts upon the beam in the direction of its own length, and in which, consequently, no part of N_1 can be independent of the tangential stresses T_2 and T_3; hence $\Psi(y, z) = 0$ or c, according as the origin of co-ordinates is at a section of no flexure or not.

Again, differentiate equation (10) in respect to y, there results

$$\frac{dN_1}{dy} = -x\left[f_y''(y, z) + \frac{d(F_z'(y, z))}{dy}\right] + \frac{d\Psi(y, z)}{dy} \quad . \quad . \quad . \quad . \quad (11)$$

In this equation any value of z may be assumed while y is considered the only variable. Let such a value for z be assumed that the equation will apply to the neutral surface. It will not destroy the force of the reasoning to suppose that surface plane, for if it is not plane the equation of its trace on the plane of normal section of the beam will be $z = f(y)$.

Now, in the neutral surface $N_1 = 0$, $T_3 = 0$ and $F_z'(y, z) = 0$ since T_2 has there its maximum value. Consequently $\frac{dN_1}{dy} = 0$, $f_y''(y, z) = 0$,

and
$$\frac{d\Psi(y, z)}{dy}\,dy = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Next, differentiate equation (10) in respect to z, and there results

$$\frac{dN_1}{dz} = -x\left[\frac{d(f_y'(y, z))}{dz} + F_z''(y, z)\right] + \frac{d\Psi(y, z)}{dz} . \quad . \quad . \quad . \quad . \quad (13)$$

Since z is considered the only variable, such a value for y may be taken that the equation will refer to that portion of a normal section of the beam which lies along the axis of symmetry of the section, for which $f'(y, z) = 0$.

Hence $$\frac{dN_1}{dz} = -xF_z''(y, z) + \frac{d\Psi(y, z)}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now $\frac{dN_1}{dz}$ is always a positive quantity, but the function $\Psi(y, z)$ is perfectly arbitrary, and it may be given such a value and sign, if it has real existence as a function of the two variables y and z, that the second member of equation (14) may have a sign contrary to that of its first member, whatever may be the value of $-xF_z''(y, z)$.

In order that equation (14) may be a true one, therefore $\frac{d\Psi(y, z)}{dz}dz = 0$; consequently

$$\frac{d\Psi(y, x)}{dy}dy + \frac{d\Psi(y, z)}{dz}dz = 0; \text{ or, } \Psi(y, z) = c, \quad . \quad . \quad . \quad (15)$$

c being a constant quantity. In the case where the origin is taken at a section of no flexure $c = 0$. Otherwise, at the free ends of a beam, and at sections of contra flexure, there will exist normal stresses parallel to the axis, since a portion of the expression for N_1 would then be independent of x.

There is then established the important equation, when the origin is taken at a section of no bending,

$$N_1 = -x\,[f_y'(y, z) + F_z'(y, z)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is seen by this equation that N_1 varies directly as x. But in this equation there is apparently involved the condition that one external force only is acting at the distance x from the section under consideration. This arises from the fact that the external forces are assumed to produce no compression at their points of application. It does not affect, however, the generality of the equation, for the last two of equations (3) show that whatever may be the bending moment, the above assumption simply means, it is so produced that the total shearing in any section is equal to that in any other, since $\frac{dT_2}{dx}$ and $\frac{dT_3}{dx}$ both equal zero.

The magnitude of the external force then, is a matter of indifference, only it must be constant for the same beam with any given system of loading.

The normal intensity N_1 is, consequently, proportional to the variable lever arm x of any given constant force which may produce the bending moment to which the

beam is subjected at the section considered; or, in other words, it is simply proportional to the bending moment.

This gives at once a method of expressing N_1 in terms of the bending moment of the external forces, and it will be sometimes convenient hereafter to use it.

Hereafter, also, unless otherwise stated, N, instead of N_1, will be written for the general value of the intensity of the normal stress parallel to the axis.

Let n and M_1 represent the values of N and the external bending moment respectively for any given section, and M the general value of the external bending moment, then, by the principle just stated,

$$N = n\frac{M}{M_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is a perfectly general expression whatever may be the position of the origin of co-ordinates.

It will now be necessary to return to the discussion of the general form of equation (16),

$$N = -x\,[f'_y(y, z) + F'_z(y, z)] + c \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

taken in connection with equations (8) and (9).

The functions $f(y, z)$ and $F(y, z)$ are perfectly arbitrary; hence it is sufficient for equilibrium to assign any laws whatever for the variations of the intensities T_2 and T_3, and when T_2 and T_3 are known N at once results from equation (18). There are not, therefore, a sufficient number of equations founded on the principles of statics to insure a solution of the problem. The "Principle of Least Resistance," however, furnishes the wanting condition. Now whatever may be the laws governing the quantities N, T_2 and T_3 there are two conditions which must be fulfilled, *i. e.* the moment of the internal stresses in any section must be equal to the moment of the external forces for the same section, and the total shearing stress in any normal section must equal the sum of the external forces acting on one side of that section. But the second of these conditions is really involved in the first, as will now be shown.

Let $f(z', y') = 0$ be the equation of the perimeter of a normal section of the beam, and $A = \int\int dzdy$ its area. Then, remembering that the coefficient of elasticity for tension is assumed equal to that for compression, the equation

expressing the equality between the moment of the internal stresses of any section, and that of the external bending forces will be

$$2\int_b^{z_1}\int_a^{y'} Nz_2dzdy = -2x\int_b^{z_1}\int_a^{y'} [f'_y(y, z) + F'_z(y, z)]\, z_2dzdy$$
$$+ 2c\int_b^{z_1}\int_a^{y'} z^2dzdy = \frac{1}{2}M. \quad \ldots \ldots \quad (18)$$

In this equation z_2 is written for convenience for $(z-b)$, and z_1 represents the maximum value of z. Of course b is the value of z for the neutral surface, and a is the value of y for the vertical axis of symmetry.

The lower limits a and b are taken so that the integration will cover one-fourth of the section, and the resulting moment in the second member will, therefore, be one-half the whole bending moment. Since the axis of z is parallel to the axis of symmetry of section, and since the external forces act parallel to it, the integral $\int\int T_2dydz = \Sigma P$, the sum of the external forces which produce the bending, while $\int\int T_3dydz = 0$; these integrals are supposed to cover the whole section.

Now $\int\int_a^{y_1} f'_y(y, z)\, z_2dzdy = \int (T_3)_a^{y'} z_2dz$; but, considering that part of the section on one side of that axis of symmetry which is parallel to the axis of z, for every positive value of z between the limits of z_1 and b there is also a negative value on the other side of the neutral surface. Hence $\int (T_3)_a^{y'} z_2dz = 0$, and the first term of the second member of equation (18) may be omitted. Again, applying the integrals to the whole surface, $\int\int z_2dzdy$ is simply the statical moment of the surface about an axis passing through its centre of gravity, consequently it is equal to zero, and the last term of the second member of equation (18) may be omitted. Hence

$$2\int_b^{z_1}\int_a^{y_1} Nz_2dzdy = -2x\int_b^{z_1}\int_a^{y'} F'_z(y, z)\, z_2dydz = \frac{M}{2}. \quad \ldots \quad (19)$$

$\int F'_z(y, z)\, z_2dz = z_2F(y, z) - \int F(y, z)\, dz$. When $z = z_1$, $F(y, z) = 0$, and when $z = b$, $z_2 = 0$. Consequently $\int_b^{z_1} F'_z(y, z)\, z_2dz = -\int_b^{z_1} F(y, z)\, dz$ and

$$\frac{1}{2}M = 2x\int_b^{z_1}\int_a^{y'} F(y, z)\, dzdy. \quad \ldots \ldots \quad (20)$$

Equation (20) shows that, *if for any section the bending moment remains the same, the shearing force also will remain constant,* which was to be proved.

If equation (20) be differentiated in respect to x, there results

$$\frac{dM}{dx} = 4\int_b^{z_1}\int_a^{y'} F(y, z)\, dzdy, \quad \ldots \ldots \quad (21)$$

which shows that the first differential coefficient of M in respect to x is equal to the total vertical shearing stress in the section, or the sum of all the external forces acting on one side of the section.

This principle, consequently that involved in equation (20), might have been determined from the fundamental equations of statics.

Now referring to equation (16), on account of the arbitrary character of the functions $f(y, z)$ and $F(y, z)$ the sum of all the *internal stresses developed in any section may have any value whatever* without effecting the equilibrium between the internal and external moments. But the principle of least resistance asserts that *the sum of all the internal stresses developed in any section shall be the least possible consistent with the imposed conditions of equilibrium.*

The only imposed conditions of equilibrium are the constancy of the total shearing or tangential stresses developed in any normal section, and the bending moment of the normal internal stresses about an axis perpendicular to the direction of those tangential stresses. But it has already been shown that the two conditions are equivalent to each other when all the external forces are vertical in direction, the axis of z being vertical also; and when the shearing stresses T_2 and moment about the axis of y are considered.

The equations of condition for the shearing stresses T_3 and moment about the axis of z will be $\int\int T_3dydz = 0$ and $\int\int Ny_2dydz = 0$. But these are simply special cases of the general equations $\int\int T_3dydz = \Sigma P$ and $\int\int Ny_2dydz = M$, consequently the reasoning applied to equation (18) will bear out the same deductions in this case. The two conditions of equilibrium are therefore involved in the latter equation in both cases.

The problem which now presents itself, therefore, is to find the law governing the intensity N so that there may be the two conditions

$$\int_b^{z_1}\int_a^{y'} Ndydz = \text{minimum}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$4\int_b^{z_1}\int_a^{y'} Nz_2dydz = M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The moment M is, of course, constant for any section while N is a variable function of y and z only, as x, like M, is constant for any section. The equations (22) and (23) may be considered typical since $y_2 = y - a$ may be written for z_2 in equation (23).

If α and t denote two variable parameters, Φ, ϕ and ψ different functions, there may be written generally

$$N = \Phi(\alpha, t), \quad y = \phi(\alpha, t) \text{ and } z = \psi(\alpha, t). \quad . \quad . \quad . \quad . \quad . \quad (24)$$

But in the case under consideration y and z are perfectly independent variables, hence the equations (24) reduce to

$$N = \Phi(a, t),\ y = \phi(a) \text{ and } z = \psi(t). \quad . \quad . \quad . \quad . \quad (25)$$

Consequently the minimum value of the quantity $\int_b^{z_1}\int_a^{y'} Ndydz$ will be found by first considering one variable constant and then the other; or in other words by first considering N a function of y and then of z, or *vice versa.* The equations (22) and (23) then become, when z is considered the only variable,

$$\int_b^{z_1} Ndz = \text{minimum}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\int_b^{z_1} Nz_2dz = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

At the neutral surface $N = 0$ and when $z = z_1$ let $N = N_0$; then (26) may take the form

$$\int_b^{z_1} Ndz = N_0z_1 - \int_b^{z_1} N'zdz = \text{minimum, in which } N' = \frac{dN}{dz}.$$

Now finding the minimum value of $\int_b^{z_1} Ndz$ is the same as finding the least value of N_0 when $\int_b^{z_1} Ndz$ is a constant quantity; the conditional equation (27) holding in both cases. Hence, putting $C = \int_b^{z_1} Ndz$, the problem involved in (26) takes the form

$$N_0 = \frac{C}{z_1} + \int_b^{z_1} N'\frac{z}{z_1}dz = \text{minimum}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Since $\frac{C}{z_1}$ is a constant quantity, the last term of the second member of the above equation is all that need be taken into consideration. If a' is a constant, then let S denote the integral of which the absolute minimum is to be found. This function S is obtained by the principles of the Calculus of Variations, by multiplying the conditional equation (27) by a' and adding the result to the variable part of equation (28). These operations give

$$S = \int_b^{z_1}\left[N'\frac{z}{z_1} + a'N(z-b)\right]dz. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The methods of the Calculus of Variations must be applied to this definite integral in order to determine the character of the function N which will give it its least value. The following system of notation is that used in the work of J. A. Serret on the Calculus:

$$V = N'\frac{z}{z_1} + a'N(-zb), \quad Z = \frac{dV}{dz} = \frac{N'}{z_1} + a'N,$$

$$Y = \frac{dV}{dN} = a'(-zb), \qquad Y' = \frac{dV}{dN'} = \frac{z}{z_1}.$$

The condition for a minimum is the following:

$$Y - \frac{dY'}{dz} = 0, \text{ or } Y = \frac{dY'}{dz}.$$

Now, since $dz = d\,(z - b)$, there results for the complete differential of V

$$dV = Zd(z-b) + YdN + Y'dN'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

But if this equation be integrated, it is evident the determinate part of the integral of the first term of the second member will be equal to that of the first member, hence

$\int (YdN + Y'dN') = c$; or, from the conditions for a minimum,

$$\int (N'dY' + Y'dN') = c;$$

$$\therefore\ N'Y' = c = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Since $\qquad Y' = \dfrac{z}{z_1}$ and $N = \dfrac{dN'}{dz}$,

$$dN = z_1 c \frac{dz}{z} \quad \therefore N = z_1 c \log z + c'. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence the curve representing the law of variation of N is a logarithmic one, and, since the value of z for the neutral surface is b, if b is taken equal to unity, c' will be zero for this case. The value for N, therefore, for a vertical plane passing through the axis of the beam will be

$$N_1 = z_1 c \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The symbol "*log*" refers, of course, to Naperian logarithms. Now, for any part of the beam z_1 must be replaced by z', since $f(z', y') = 0$ is the equation of the perimeter of the section. Equation (33), therefore, for any strip of elements parallel to the axis of z and at any distance y from the origin, will take the form

$$N = z'c \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Let N_0' represent the value of N for any point of the perimeter of the section, except that one for which $z = e_1$; and for that point let N_0 be written, it will represent the greatest intensity of stress in the section. When $z = z'$, $N = N_0'$; consequently, for equation (34), $c = \dfrac{N_0'}{z' \log z'}$,

$$\therefore N = \frac{N_0'}{\log z'} \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

When $z = z_1$, $\;N_1 = N_0$, $\quad \therefore c = \dfrac{N_0}{z_1 \log z_1}$ for equation (33), and

$$N_1 = \frac{N_0}{\log z_1} \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

From what has already been said, in regard to the general equations of condition for moments and shearing stresses in horizontal planes, between equations (21) and (22), it is evident that the same steps precisely would have to be taken in order to determine the law of variation of N with y, as were taken to determine the connection between N and z. In fact, since y and z are considered variable only in turn, y may be written for z in the general operations for determining the least value of the definite integral S. Hence the typical equation for N may be written in terms of y,

$$N = \frac{N_1}{\log y_1} \log y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The quantity y_1 denotes the half width of the beam added to a. But, as was done in the case of z, a is assumed to be equal to unity in determining the constant $\frac{N'}{\log y_1}$.

Now, in writing the equation (37) there is virtually assumed to be a surface of no stress of the kind N at the distance $(y_1 - 1)$ from the vertical axis of symmetry of the section. In other words, referring to Fig. 1, O' is really the assumed origin and HK the supposed position of the axis of z, while the surface of no stress touches the beam at m. The greatest value of N, therefore, in any horizontal plane, is N_1 found in the vertical axis of symmetry of the beam. The point O is at the distance unity on one side of, and below, the centre C of the section; and it is most convenient to take that point for the origin of co-ordinates. $O'O$ is equal to $(y_1 + 1)$ and $O'F$ is the y of equation (37). This latter quantity in terms of OF, the new y, will be $(y_1 + 1 - OF)$ $= (y_1 - y + 1)$. Consequently, equation (37) takes the form

$$N = \frac{N_1}{\log y_1} \log (y_1 - y + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

When $y = BD = y'$ then $N = N_0'$,

$$N_0' = \frac{N_1}{\log y_1} \log (y_1 - y + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

N_1 is determined by equation (36), but z' must be written for z in that equation, then

$$N_1 = \frac{N_0}{\log z_1} \log z', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\therefore \quad N'_0 = \frac{\log z'}{\log z_1} \frac{N_0}{\log y_1} \log (y_1 - y' + 1). \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Substituting in equation (35), there results

$$N = \frac{N_0}{\log z_1} \frac{\log z}{\log y_1} \log (y_1 - y' + 1) \text{. . . . . .} \quad (42)$$

This gives the general value of N in terms of the greatest intensity N_0 of the entire section. If $z = z'$, then, by referring to equations (38) and (40) it is seen that $N = N'_0$. If $z = 1$ the equation (42) refers to the neutral surface and $N = 0$.

If $y' = 1$, then the vertical axis of symmetry is referred to, and

$$N = N_1 = \frac{N_0}{\log z_1} \log z.$$

Before passing on farther in the analytical discussion of the problem, it will be well to consider the form of the double curved surface which represents graphically the law of variation of the intensity N.

The closed curve in Fig. 1 represents a normal section of the beam, O being the origin of co-ordinates. Now if normal lines be drawn at each point of the section of Fig. 1 whose lengths represent intensities, N, at the different points, a double curved surface will enclose their extremities from which logarithmic curves, represented by the equations already given, will be cut by vertical and horizontal planes. The shaded portion of Fig. 2 represents a section cut by a vertical plane passed through the axis of the beam, and equation (36) is the equation to its perimeter. The shaded portion of Fig. 3 is a horizontal section made by a plane passed through RS, Fig. 1; the general equation for which is equation (38). CD of Fig. 3 is equal to FH of Fig. 2. All vertical planes will cut sections similar to that in Fig. 2; these sections will have for their equation, equation (35). All horizontal planes will cut sections similar to that in Fig. 3 and equation (38) will be the general equation to their perimeters.

The tangent of the angle made by the curve at C, Fig. 2, with BC is equal to $\frac{dN_1}{dz} = \frac{N_0}{\log z_1}$, since for that point $z = 1$. The general value for the tangent is $\frac{dN_1}{dz} = \frac{N_0}{\log z_1} \frac{1}{z}$. Hence the curve becomes parallel to BC at an infinite distance from the origin, and has a horizontal asymptote passing through the origin. The same reasoning applies to the curves of the other sections.

$\frac{d^2N}{dz^2} = -\frac{N_0}{\log z_1} \frac{1}{z^2}$; hence the vertical curves are concave towards the axis of

z. For the same reason the horizontal curves are concave towards the axis of y. The whole surface therefore is concave towards the plane of section.

Nothing has been said in regard to the determination of the position of the neutral surface, except the statement made in the beginning, which would make it a plane before flexure passing through the centre of gravity of a normal section, on the supposition that the coefficients for tension and compression are equal to each other. The true principle has so long been recognized that it is not necessary to speak farther of it here.

Referring to equations (17) and (42), it is evident that the general value for the intensity N will be

$$N = \frac{M}{M_1} \frac{N_0}{\log z_1} \frac{\log z}{\log y_1} \log (y_1 - y' + 1) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

It is also evident that equation (43) may be so written as to apply to a horizontal plane at the distance z' from the origin; it will then take the form

$$N = \frac{M}{M_1} \frac{N_0}{\log z_1} \frac{\log z'}{\log y_1} \log (y_1 - y + 1) \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Although this is deduced immediately from equation (43), it may be demonstrated in precisely the same manner as was that equation.

The moment of resistance of the beam may now be easily written, though the integration involved may yet be found impossible in some cases and intricate in all but rectangular beams.

It is well known that the tangential stresses existing on the sides of a small parallelopipedical portion of any material constitute a system of forces in equilibrium. Consequently, the moment of resistance in any section will be the sum of the small moments $Ndydz \, . \, (z - 1)$. The lever arm of each of the small forces $Ndydz$ is $(z - 1)$, because the centre of moments is taken in the neutral surface and the origin of co-ordinates is at the distance unity below that surface.

Since the normal sections of all the beams considered are symmetrical and without re-entrant outlines, the following equation at once results:

$$\frac{1}{4} M = \frac{N_0}{\log z_1 \log y_1} \int_1^{y_1} \int_1^{z'} \log (y_1 - y' + 1) \log z \, . \, (z - 1) \, dzdy'. \qquad (45)$$

N_0 is, of course, the greatest intensity in the given section. Since y is an independent variable, dy may be taken equal to dy'.

Now, $\int_1^{z'} (z - 1) \log z \, . \, dz = \frac{1}{2} z'^2 \log z' - \frac{1}{4} z'^2 - z' \log z' + z' - \frac{3}{4}$, and

$z' = f(y')$ is the equation to the perimeter of the section. Consequently

$$\frac{1}{4}M = \frac{N_0}{\log z_1 \log y_1}\int_1^{y_1}\left(\frac{1}{2}\overline{f(y')}^2 \log f(y') - \frac{1}{4}\overline{f(y')}^2 - f(y')\log f(y') + f(y') - \frac{3}{4}\right)\log(y_1 - y' + 1)\,dy'. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

This intricate expression reduces to a much simpler one for beams of rectangular section. Equation (45) might have been written in terms of z' and y, in which case, equation (46) would have been found in terms of z', but would not be in as convenient shape.

The general values of the displacements u, v and w may now be approximately determined. It must be remembered that these displacements will only exist when N_2, N_3 and T_1 are each equal to zero. From the equation (5) and (6) there results the relation

$$\frac{dv}{dy} = \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Then, from either equation (5) or equation (6),

$$\frac{du}{dx} = -\frac{2(\lambda + \mu)}{\lambda}\frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

But, from equation (1),

$$N = \lambda\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + 2\mu\frac{du}{dx} = 2\lambda\frac{dw}{dz} + (\lambda + 2\mu)\frac{du}{dx}. \quad . \quad . \quad (49)$$

Substituting from (48) in (49), there results

$$N = -\frac{2\mu}{\lambda}(3\lambda + 2\mu)\frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From equation (50), in connection with equation (47), there at once result

$$w = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\int N dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$v = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\int N dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Equation (50) gives also the relation

$$-\frac{2(\lambda + \mu)}{\lambda}\frac{dw}{dz} = \frac{(\lambda + \mu)}{\mu(3\lambda + 2\mu)}N = EN.$$

Combining this with equation (48), there results

$$u = E\int N dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The coefficient of elasticity $E = \dfrac{\lambda + \mu}{\mu(3\lambda + 2\mu)}$ is written as M. Lamé uses it, *i. e.* so as to represent the strain for each unit of stress. For wrought iron

E would have an average value of, say, $\frac{1}{26000000}$. The relation between E, μ and λ will be found given in the work of M. Lamé before mentioned. In finding the value for w, N is to be taken in terms of y' and z, and in determining v, it is to be taken in terms of z' and y. Substituting the values for N, there result the following expressions for u, v and w:

$$w = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1)(z \log z - z) + f(x, y),$$

$$v = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log z' [(y_1 - y + 1) - (y_1 - y + 1) \log (y_1 - y + 1)] + f(x, z),$$

$$u = \frac{EN_0}{M_1 \log z_1 \log y_1} \log (y_1 - y' + 1) \log z \int M dx + f(z, y).$$

The functions $f(\;)$ must be added in each integration because w, v and u are each functions of the three independent variables x, y and z.

Let Δ be the deflection of the upper surface of the beam at *any* point; then when $z = z'$, $w = \Delta$. In the vertical plane of symmetry for the beam $v = 0$; hence when $y = 1$, v will equal zero.

The term $f(z, y)$ in the expression for u will depend upon the configuration of that section of the beam in which the origin of co-ordinates is located, it expresses the displacement in the direction of x for that section. If that section remains plane and vertical after flexure $f(z, y)$ will reduce to zero, or a constant, and it will always be equal to zero for the neutral surface if it be assumed that the section containing the origin suffers no movement, as a whole during flexure. For any other point not in the neutral surface its value will depend on the distribution of tangential stress in the section where the origin is found, and its value is not easy to determine. In all cases of ordinary experience it is a very small quantity compared with the other parts of the deflection, and essentially no error will be committed by its omission; such an omission will be made in equation (56).

By introducing the given conditions the values of w, v and u will be written as follows:

$$w = \frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1)(z' \log z' - z' - z \log z + z) + \Delta, \quad (54)$$

$$v = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log z' [y_1 \log y_1 + (1 - y) - (y_1 - y + 1) \log (y_1 - y + 1)], \quad (55)$$

$$u = \frac{EN_0}{M_1 \log z_1 \log y_1} \log (y_1 - y' + 1) \log z \int \Sigma P (x_1 - x)\, dx. \quad (56)$$

Now, equation (54) has been written involving Δ, the deflection of the upper surface of the beam, but it must be remembered that w in the values of N, T_2 and T_3 represents simply the displacements in the given section, or that which is caused by the stresses acting and not by any bodily movement of any portion of the beam. In writing the value of T_2, therefore, in equation (57), Δ must be omitted in equation (54). Otherwise, it would be true, as a general principle, that the shearing stress in any section is dependent on the deflection Δ, which is evidently not true. In equation (56), x_1 is the co-ordinate of the section under consideration, and x is the general value of the abscissa of the point of application of the force P; or, in other words, $M = \Sigma P\,(x_1 - x)$.

It is seen from the value of w immediately preceding equation (54), that the deflection of the neutral surface at any point is independent of the variable z, and is a function of the independent variables x and y. This result shows that the neutral surface is not a cylindrical one after flexure, although it is symmetrical in reference to a vertical plane of symmetry for the beam. The neutral surface, then, is a surface of double curvature for all beams except those with rectangular sections, for which it is cylindrical.

Since Δ depends on x and y, and not on z, the deflection of the neutral surface may be determined if the maximum intensity of direct stress N_0 is known for the given section, as will be seen hereafter.

In equations (1) there are given general values for the intensities of the tangential stresses T_2 and T_3 in terms of u, v and w. Using equations (54–56), the two following equations are deduced, remembering what has already been said in regard to equation (54):

$$T_2 = \frac{M_1 \log z_1 \log y_1}{\log(y_1 - y' + 1)} \frac{2(3\lambda + 2\mu)}{\lambda N_0} (z' \log z' - z' - z \log z + z)\, \Sigma P$$
$$+ \frac{(\lambda + \mu)}{(3\lambda + 2\mu)} \frac{N_0}{M_1 \log z_1 \log y_1} \frac{\log(y_1 - y' + 1)}{z} \int \Sigma P\,(x_1 - x)\,dx\,, \quad . \quad . \quad . \quad (57)$$

$$T_3 = -\frac{\lambda N_0}{2(3\lambda + 2\mu)} \frac{\log z'}{M_1 \log z_1 \log y_1} [y_1 \log y_1 + (1 - y) - (y_1 - y + 1)\log(y_1 - y + 1)]\, \Sigma P$$
$$- \frac{\lambda + \mu}{(3\lambda + 2\mu)} \frac{N_0 \log z'}{M_1 \log z_1 \log y_1} \frac{\int \Sigma P\,(x_1 - x)\,dx}{(y_1 - y + 1)} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

It has been assumed that $f\,(z, y)$ in the value of u is equal to zero for both equations (57) and (58). If this cannot be admitted, then $\mu \dfrac{df(z, y)}{dz}$ is to

be added to the second member of equation (57), and $\mu \frac{df(z,y)}{dy}$ to that of equation (58).

If the partial differential coefficients of T_2 and T_3 be taken in respect to z and y, the two following equations will result after having substituted from the general values of N:

$$\frac{dT_2}{dz} = -\frac{(\lambda+\mu)}{(3\lambda+2\mu)} \cdot \frac{\int Mdx}{M} \cdot \frac{d^2N}{dz^2} - \frac{\lambda}{2(3\lambda+2\mu)} \frac{\Sigma P}{M} N, \quad . \quad . \quad . \quad (59)$$

$$\frac{dT_3}{dy} = \frac{(\lambda+\mu)}{(3\lambda+2\mu)} \cdot \frac{\int Mdx}{M} \cdot \frac{d^2N}{dy^2} - \frac{\lambda}{2(3\lambda+2\mu)} \frac{\Sigma P}{M} N. \quad . \quad . \quad . \quad . \quad (60)$$

Now, from equation (16), it is seen that:

$$\frac{dN_1}{dz} = \frac{\Sigma P}{M} N = -\left(\frac{dT_2}{dz} + \frac{dT_3}{dy}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

But equation (60) shows that equation (61) is only true when $\lambda = -\mu$ or $\mu = -\lambda$; or when $E = 0$, since $E = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}$; or when the material is rigid so far as tensile and compressive stresses are concerned. Lateral displacements due to shearing stresses, however, may be supposed to exist.

Equations (1) give the general values of the intensities N_1, T_2 and T_3, but in order that equilibrium may exist they must be subject to the conditions of equations (4), which are perfectly independent of the equations (1). In fact equations (4) are founded on the first principles of statics and are perfectly independent of the nature of the material in which stress may exist. This matter will be specially noticed farther on.

The equations (59), (60) and (61) show that the distribution of the shearing or tangential stresses in the beam subjected to flexure is independent of the quantities λ and μ, and is the same whether the beam be supposed rigid or elastic with a finite value of E. Making $\lambda = -\mu$ therefore in equations (57) and (58), there results

$$T_2 = \frac{N_0}{2M} \frac{\log(y_1 - y' + 1)}{\log z_1 \log y_1} (z' \log z' - z' - z \log z + z) \Sigma P, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$T_3 = -\frac{N_0}{2M_1} \frac{\log z'}{\log z_1 \log y_1} [y_1 \log y_1 + (1-y) - (y_1 - y + 1) \log(y_1 - y + 1)] \Sigma P. \quad . \quad (63)$$

These are the true values of the intensities of the tangential stresses, and it will hereafter be shown that $4\int_1^{y_1}\int_1^{z'} T_2 dz dy = \Sigma P$, as should be the case.

It has already been shown that $\frac{dM}{dx} = \Sigma P$, consequently equations (62) and (63) may be written in terms of $\frac{dM}{dx}$, and it will sometimes be convenient to use them in that form hereafter.

There is an apparent anomaly in the fact that equations (57) and (58) are the expressions derived directly from the general values of T_2 and T_3 in equations (1), while equations (62) and (63) are the true values of these intensities. The explanation is found in what has already been said in regard to the intensities being the same as in a material for which E is equal to zero. Equations (62) and (63) also show that T_2 and T_3 are independent of x, except in so far as that variable may enter the summation ΣP, which is consistent with one of the first general equations of condition.

From equations (62) and (63) the following results flow: if $z' = 1$, $T_3 = 0$ for all values of y; if $z = z' = 1$, $T_2 = 0$, and if $z = z'$ only, $T_2 = 0$; if $y = 1$, $T_3 = 0$. These results are as they should be, and might have been anticipated.

Another method of deducing the displacement w, in which Δ will represent the deflection of any point of the neutral surface, is the one which follows. It is somewhat more convenient in the treatment of beams with rectangular cross sections. Let Δ then represent the deflection of any point of the neutral surface. When $z = 1$, in the value of w immediately preceding equation (54), $w = \Delta$, hence

$$f(x, y) = \Delta - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1). \quad . \quad . \quad (64)$$

If Δ', therefore, represents the general value of the deflection, there will result, instead of equation (54),

$$\Delta' = - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1) \,.\, (z \log z - z + 1) + \Delta. \quad (65)$$

Now, let Δ'' represent the value of Δ' when $z = z'$, then the quantity w, which is to be used in writing the value of T_2, will be equal to $\Delta' - \Delta''$. Hence

$$w = \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1) \,.\, (z' \log z' - z' - z \log z + z). \quad (66)$$

Equation (66) is the same as equation (54) with Δ omitted from the latter. The consequences indicated by equations (56) and (57) might be deduced more simply, perhaps, from equations (65) and (66) than from equation (54) and the one preceding it.

It is to be noticed here that the expressions for the intensities N and T_2, as well as T_3, are perfectly general, although the original equations of condition were based on the supposition that the bending moment should be produced by a single force or a couple. Their generality is due to two facts: a given amount of shearing stress will always be distributed over the same section in precisely the same way, whether that amount is made up of reaction at a point of support combined with external loads imposed between that point and the given section, or whether that amount is equal to a single force P hung at the free end of a beam; and a bending moment M may be produced by a single force P or by a number of forces whose combined effect produces the given moment, and the distribution of the direct stresses of tension and compression will be precisely the same in each case.

By reference to equation (46) it is seen that the quantity $\frac{N_0}{M}$, or $\frac{N_0}{M_1}$ (the intensity and the moment must belong to the same section), is not altogether dependent on the form of the cross section, since the quantities $\log z_1$, $\log y_1$ and $\frac{3}{4}$ enter the expression for M, *but is a constant quantity for the same beam.* In like manner $\frac{N}{M}$ is constant for the same beam, if N is always taken at a point whose co-ordinates y and z are the same in the different sections.

It is evident that the maximum value of T_2 will be found at the centre of any section; consequently its value will be determined by making $z' = z_1$, $y' = 1$ and $z = 1$ in equation (62). Denoting the maximum value of T_2 by T_m, there results

$$T_m = \frac{N_0}{2M_1} \frac{1}{\log z_1} (z_1 \log z_1 - z_1 + 1) \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Let A be the area of the section of the beam to which equation (67) applies, then the mean shearing intensity in any section will be $\frac{\Sigma P}{A}$. The ratio, therefore, between the maximum and mean intensities of shear in any section will be

$$T_m \frac{A}{\Sigma P} = \frac{N_0}{2M_1} \frac{(z_1 \log z_1 - z_1 + 1)}{\log z_1} A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This expression is not constant for the same form of cross section, but is constant for the same beam.

When ΣP is equal to zero, both T_2 and T_3 reduce to nothing. This case exists where a portion of a beam is bent by a couple and where evidently the

curve of flexure must be circular, since N cannot vary if T_2 and T_3 are both equal to zero, as equation (10) shows. This is one of the special cases in which $\Psi (y, z)$ is a constant.

The expressions for T_2 and T_3 show how the shearing stress is supposed to be distributed at the free ends of beams and at sections of contraflexure, and furnishes the data for determining the quantity $f(z, y)$ in the value for the longitudinal displacement u. As, however, it is of little practical value it will not be determined. The reaction, therefore, at the free ends of beams and external forces acting at sections of contraflexure are supposed to be so distributed over the sections of the beam that $\int\int T_2 dydz = \Sigma P$.

The deflection of the beam is next to be determined, and it has already been shown that that part of it, Δ, due to the bodily movement of a portion of the beam is not a function of z, but is dependent only on x and y. It varies of course with the half depth of the beam, or with what amounts to the same thing, the quantity z_1.

The movement of the molecules of the material, relatively to each other, in any given section, is to be determined by the value of w from equation (66) which was used in fixing the value of T_2.

Let Δ_1 represent the deflection of any point of the *upper surface* of the beam. From what has already been said in regard to Δ, and, from the general conditions of the problem, it is clear that this depends only on the lengthening or shortening of the exterior fibres in the upper surface of the beam.

The upper surface of the beam is here mentioned, although "the lower surface" might have been written just as well.

The rate $\frac{du}{dx}$ of the longitudinal displacement at any point, is due to the intensity N, or N_0', if that point is in the exterior surface. Let u_0 be the value of u for any point where N_0' exists, then $\frac{du_0}{dx} = EN_0'$. The coefficient of elasticity E, of course, represents the rate of lengthening or shortening of a fibre at any point for each unit of N_0'. Now, if the beam be divided into indefinitely thin rectangular portions by vertical planes parallel to the axis of the beam, each portion may be supposed to be an actual rectangular beam subjected to such a moment that the greatest intensity of direct stress is equal to N_0' at the given section. The sum of all these elementary moments for

any section will be equal to the moment to which the original beam is subjected; and the sum ΣP of all the external forces acting on all the elementary beams for the same sections, will be equal to the sum ΣP for the original beam at the same section. From this it follows that the deflections of different points in the exterior surface have different values; also, that the deflection of any such point is precisely the same as that which a rectangular beam would have if the circumstances of loading and length were the same in each case, and if the depth of the given beam at the given point were equal to the depth of the supposed rectangular beam; which conditions make N_0' the same for each.

These considerations show that the deflection of that point in the exterior surface of a beam which is farthest from the neutral surface, is independent of the form of cross section, and is the same as that of a rectangular beam in the same circumstances; which results also from the "common theory."

In Figure 5, let AB be a portion of the line of intersection of a longitudinal plane with the neutral surface, and C, the centre of curvature of $AB . BD$ is parallel to AC, then $FD = AB = 1$. Let $AC = r$, then will $DE = \frac{du_0}{dx}$. From similarity of triangles, since $AF = BD = (z' - 1)$, in general

$$\frac{\frac{du_0}{dx}}{z'-1} = \frac{1}{r} = \frac{EN_0'}{z'-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

This gives the value of the reciprocal of the radius of curvature in a longitudinal plane at any point, and its general form and method of demonstration is precisely that used in the common theory. If there be written that approximate value of $\frac{1}{r} = \frac{d^2y}{dx^2}$, which was introduced by Navier, and $N_0' = f(M)$, there will result

$$\frac{d^2y}{dx^2} = \frac{Ef(M)}{(z'-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The y in equation (70) is not the one heretofore used, but represents the deflection due to the displacement EN_0', and taken between the proper limits, is equal to Δ_1.

Before developing this matter of the deflection farther, it will be well, for the reasons already given, to find equations for beams of rectangular sections.

In order to make the general value of N in equations (43) or (44) apply to rectangular beams, it is only necessary to put y or y' equal to unity and write z for z'. Performing these simple operations, there will result

$$N = \frac{M}{M_1} \frac{N_0}{\log z_1} \log z \,. \quad \ldots \ldots \ldots \quad (71)$$

The same substitutions made in equations (62) and (63) give

$$T_2 = \frac{N_0}{2M_1} \frac{(z_1 \log z_1 - z \log z - z_1 + z)}{\log z_1} \Sigma P \,, \quad \ldots \ldots \quad (72)$$

$$T_3 = 0 \,, \quad \ldots \ldots \ldots \ldots \ldots \quad (73)$$

The result shown in equation (73) was to have been anticipated.

Equations (71) and (72) might have been established directly by a course of reasoning precisely similar to that followed for a beam of any symmetrical but solid section, in which case, in addition to equations (5), (6) and (7), there would have been the one indicating that $T_3 = 0$.

Let b be the breadth of the rectangular beam, then equation (45) will reduce to the form

$$M = 2b \frac{N_0}{\log z_1} \int_1^{z_1} (z-1) \log z \,.\, dz$$

$$= 2b \frac{N_0}{\log z_1} \left[\frac{1}{2} z_2 \log z - \frac{1}{4} z_2 - z \log z + z \right]_1^{z_1}$$

$$= 2b \frac{N_0}{\log z_1} \left(\frac{1}{4} z_1^2 \log \frac{z_1^2}{e} - z_1 \log \frac{z_1}{e} - \frac{3}{4} \right) . \quad \ldots \ldots \quad (74)$$

In equation (74) e is the base of the Naperian system of logarithms, and N_0 is the greatest intensity of direct stress in the given section. The quantity $\frac{M}{b}$ is the bending moment for each unit of breadth, and it is seen from equation (74) that $\frac{bN_0}{M}$ is a constant quantity for all rectangular beams of the same depth.

The sum of all the shearing stresses in the section ought to be equal to ΣP. Hence, from the general expression following equation (63),

$$4b \int_1^{1} T_2 dz = \frac{2bN_0}{M_1 \log z_1} \left(\frac{1}{4} z_1^2 \log \frac{z_1^2}{e} - z_1 \log \frac{z_1}{e} - \frac{3}{4} \right) \Sigma P \,.$$

But by equation (74) the second member of this equation is equal to $-\Sigma P$; hence

$$4b \int_1^{z_1} T_2 dz = \Sigma P. \quad \ldots \ldots \ldots \quad (75)$$

It has already been stated that the assumption $N_3 = 0$ is no cause of error in the results for beams of rectangular section; it will next be shown

that such is the case. For rectangular beams the equations (2) reduce to the following:

$$\frac{dN_1}{dx}+\frac{dT_2}{dz}=0,$$

$$\frac{dT_2}{dx}+\frac{dN_3}{dz}=0.$$

The three intensities N_1, T_2 and N_3 are each functions of z and x only. The "principle of least resistance" determines N_1 at once as given by equation (71); T_2 at once follows in equation (72). The second of the above equations in connection with equation (72) gives

$$\frac{dN_3}{dz}=-\frac{dT_2}{dx}=-\frac{N_0}{2M_1}\left\{\frac{(z_1\log z_1-z_1)-(z\log z-z)}{\log z_1}\right\}\frac{d\Sigma P}{dx}.$$

The quantity $\frac{d\Sigma P}{dx}$ is the intensity of external vertical pressure at any point; denote it by $-p$. Then

$$N_3=+\frac{N_0 p}{2M_1\log z_1}\left\{(z_1\log z_1-z_1)\,z-\frac{1}{2}\,z^2\log z+\frac{3}{4}\,z^2\right\}+f(x).$$

When $z=z_1$, $N_3=-p$, hence

$$N_3=-\frac{N_0 p}{2M_1\log z_1}\left\{\frac{1}{2}\,z_1^2\log z_1-\frac{1}{4}\,z_1^2-(z_1\log z_1-z_1)\,z+\frac{1}{2}z^2\log z-\frac{3}{4}z^2\right\}-p.$$

These values of the intensities N_1, T_2 and N_3 satisfy the two simultaneous equations of condition given above.

The assertion which immediately follows equation (63) may now be proved without difficulty. Equation (62) may be put under the following form:

$$T_2=\frac{\Sigma P}{2M_1}\left\{\frac{N_0}{\log z'}\;\frac{\log z'\log(y_1-y'+1)}{\log z_1\log y_1}\,(z'\log z'-z'-z\log z+z)\right\};$$

or, by equation (41),

$$T_2=\frac{\Sigma P}{2M_1}\left\{\frac{N_0'}{\log z'}\,(z'\log z'-z'-z\log z+z)\right\}\quad\ldots\ldots\quad(76)$$

Now, the given beam is equivalent to an indefinite number of elementary beams of the constant or variable width $2dy$ (corresponding to b), and having the variable depth $2\,(z'-1)$. Hence, the integral $4\int\int T_2dzdy$ may be put in the following form:

$$4\int_1^{y_1}\int_1^{z'}T_2dzdy=\frac{\Sigma P}{M_1}\,\Sigma\left\{\int_1^{z'}\frac{N_0'}{\log z'}\,(z'\log z'-z'-z\log z+z)\right\}2dy.\quad\ldots\quad(77)$$

But from the equations immediately preceding equation (75) it is evident that that part of the second member of equation (77) which follows the second

Σ is the general expression for the moment to which any elementary rectangular beam is subjected; hence the sum of all those moments denoted by Σ must be equal to (M_1). Hence

$$4\int_1^{y_1}\int_1^{z'} T_2 dy dz = \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

It is evident from the preceding that the expression, as a general one, $\frac{\Sigma P}{M_1}$ is the same for all the elementary beams and the original beam itself. M_1 and ΣP belong, of course, to the same section.

The subject of deflection can now be resumed. Let Y_1 represent the definite integral in equation (46), then from equation (41) there results

$$N_0' = \frac{\log z' \log (y_1 - y' + 1)}{4Y_1} M = f(M). \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Equation (79) gives the value of $f(M)$ in equation (70) for the general case. As has already been shown, however, it may be only necessary to find the function for a rectangular section in which $b = 1$ and $z_1 = z'$. To determine, therefore, that part of the deflection which is denoted by Δ_1, find the value of N_0' from equation (79) and substitute it in equation (70), then, if Z_1 be put for $\frac{\log z' \log (y_1 - y' + 1)}{4Y_1(z'-1)}$, that equation will give

$$y = \Delta_1 = EZ_1 \int\int M dx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

This is precisely the expression given by the "common theory" if $\frac{1}{I}$ (I being the moment of inertia of the cross section) be written for Z_1. The ordinary values for y may therefore be used in equation (80) by inserting in the formulæ of the "common theory" Z_1 for $\frac{1}{I}$.

If N_0' is known for any point, then by equation (74)

$$N_0' = \frac{\frac{1}{2}\log z'}{\left(\frac{1}{4} z'^2 \log \frac{z'^2}{e} - z' \log \frac{z'}{e} - \frac{3}{4}\right)} M = ZM. \quad . \quad . \quad . \quad (81)$$

That which is represented by Z is evident from the equation. There will then result as before

$$y = \Delta_1 = EZ \int\int M dx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

The remarks following equation (80) apply also, as is evident, to equation (82).

Equations (80) and (82) give that part of the deflection which is due to the bodily moment of a portion of the beam and which is caused by the longitudinal displacement u. Another part is that due to the shearing stress T_2 at the neutral surface, which causes layers, made by vertical planes normal to the axis of the beam, to slip by each other to a greater or less extent.

It should be understood that when the "deflection of the beam" or "total deflection" is spoken of, the neutral surface is what is referred to.

That portion of the total deflection which is due to T_2, or the displacement (vertical) in any given section is given by equation (66) after making $\lambda = -\mu$. Let M_0 be the value of M at the point from which the deflection is measured, and w_1 represent this part of the total deflection, then

$$w_1 = \frac{M - M_0}{2M_1\mu} \frac{N_0}{\log z_1 \log y_1} \log(y_1 - y' + 1)(z' \log z' - z' - z \log z + z). \quad . \quad (83)$$

In many cases M_0 belongs to the free end of a beam and is equal to zero.

Equation (83) might have been determined by making use of $T_2 = g\mu$, g being the angle at any point made by the trace of a vertical longitudinal plane on the neutral surface with a horizontal line. When equations (81) and (83) refer to rectangular beams, z' becomes equal to z_1. Since w and T_2 both take the value zero for $z = z'$ it follows that the depth of the beam remains the same after flexure as before for bodies of the kind of material assumed. The lateral contractions and expansions of the material at any point are just equal to the displacements due to internal tangential stresses.

There is one other source of deflection which, however, is evidently so exceedingly small in reference to the two already mentioned, that an expression for it will not be sought, though the data given are sufficient for it. This is the curved form assumed by the free-end section of the beam. If that section remains plane and normal to the axis of the beam after flexure, as has been assumed, then $\Delta_1 + w_1$ gives the total deflection. In reality, however, each point of the end section is displaced longitudinally in consequence of the distribution of the reaction in the manner already given by the general value of T_2. This third part of the deflection is due to this displacement being supposed uniformly distributed throughout the length of the beam. Such an operation would produce deflection without causing any direct stress of the kind N. Since, however, the reaction is probably never distributed in the manner indicated (the end sections therefore remaining essentially plane)

and since this part is, at best, very small compared with the others, it will be disregarded.

The total deflection of the neutral surface will then be

$$\Delta = \Delta_1 + w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

From equation (70), combined with equation (79) or (81), according to the shape of the cross section of the beam, there result the following equations:

$$M = \frac{1}{EZ} \frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

$$M = \frac{1}{EZ_1} \frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

It follows from these equations that all the results of the "common theory" can be used in the application of the formulæ of this paper by simply writing $\frac{1}{Z}$ or $\frac{1}{Z_1}$ for I in those results. Consequently, the "Theorem of Three Moments" will retain precisely the same form as before, the change above indicated being only necessary.

It is much to be regretted that the form of Z is so complicated that in some cases the integration will probably be found to be impossible at present. The integration, however, will have to be performed but once for the same form of cross section. The quantity Z_1 for rectangular cross sections is of comparatively simple form, and, fortunately, by far of the greatest importance.

The form assumed by a rectangular cross section, when the beam is subjected to flexure, may easily be found in the usual manner, and requires no special attention here.

Although not strictly a part of this discussion, yet it may be interesting to notice under what conditions the assumed law for the intensity N, in the "common theory," is a true one. A beam of rectangular cross section will be considered. The notation will be the same as that used in the vicinity of equations (29–32.)

The equation of condition, which shows that the moment of resistance is constant, is

$$\int_a^{z_1} N(z-a)\, dz = \text{constant} = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

Now, let it be required to find what law for N will make the volume of revolution $\int 2\pi N^2 dz$ a minimum, or give the following equation

$$2\pi \int_a^{z_1} N^2 dz = \text{minimum}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Proceeding as before

$$S = \int_a^{z_1} (2\pi N^2 + a'N(z-a))\, dz\,,$$

$$\therefore V = 2\pi N^2 + a'N(z-a)\,; \quad Z = \frac{dV}{d(z-a)} = a'N;$$

$$Y = \frac{dV}{dN} = 4\pi N + a'(z-a)\,; \quad Y' = 0\,; \ \&c.$$

Hence, from the calculus of variations,

$$4\pi N + a'(z-a) = 0 \quad \therefore N = -\frac{a'(z-a)}{4\pi} \,. \quad . \quad . \quad . \quad . \quad (89)$$

The quantity a' must be such that

$$2\int_a^{z_1} -\frac{a'(z-a)^2}{4\pi}\, dz = M\,.$$

Equation (89) shows that *the intensity N varies directly as the distance from the neutral surface,* which is the law assumed in the "common theory" of flexure.

The law is, therefore, based on the erroneous equation (88); to be true, 2π should not appear in that equation, and N^2 should be replaced by N. $\frac{d^2V}{dN_2}$ is a positive constant, showing that equation (89) gives a value that will make V a minimum.

These last operations show that in all ordinary cases the logarithmic curve will not be a very great departure from a straight line.

It has been assumed that the coefficients of elasticity for tension and compression are equal to each other; it is easy, however, to determine the position of the neutral surface when they are not, for beams with rectangular cross sections. In Figure 6 let $ABFG$ represent the portion of a beam subjected to flexure, supposing the coefficients of elasticity to be equal to each other; the neutral surface DK will be half way between the exterior surfaces AB and GF. Now, let there be another beam $GHCF$ whose neutral and lower surfaces are coincident with those of the former, and let HC represent the upper surface of this second beam. The normal distance z_1 from DK to HC will bear such a relation to z_0, the normal distance from DK to GF, that the stress of the kind N, developed in that part of the section z_1, will be numerically equal (but of opposite sign) to that developed in the part z_0. Let E represent the smallest coefficient of elasticity and E_m the largest. From equation (71), making $M = M_1$ since any section may be taken, there results in general

$$\int_1^z Ndz = \frac{N_0}{\log z_0}\left(z \log \frac{z}{e} + 1\right).$$

On account of the above assumptions, EN on one side of the neutral surface must be equal to $E_m N$ at the same distance from it on the other. The equation, therefore, which shows that the algebraic total of all the normal stresses in any section is equal to zero, is

$$E \frac{N_0}{\log z_0}\left(z_0 \log \frac{z_0}{e} + 1\right) = E_m \frac{N_0}{\log z_0}\left(z_1 \log \frac{z_1}{e} + 1\right),$$

$$\therefore E\,(z_0 \log z_0 - z_0 + 1) = E_m\,(z_1 \log z_1 - z_1 + 1). \quad . \quad . \quad . \quad . \quad (90)$$

After substituting the values of E and E_m, this transcendental equation can easily be solved by trial.

Since $E_m > E$, z_1 is of course smaller than z_0 in all cases.

This completes the strictly analytical part of the discussion, but there remains to be shown that the results are perfectly general in their character.

The general equations (2) of equilibrium were established in a manner entirely independent of the nature of the material of which the body is composed. They are three linear differential relations between six functions of the three independent variables x, y and z *only*, *i. e.*, the differentiations are in respect to those variables only. The integrations will, therefore, be made in respect to the same variables, and, in order that they may be made, there must be given certain known conditions depending on the method of application of the external forces and purely mechanical principles; these conditions are evidently entirely independent of the nature of the material. The integrations being made, the six intensities N and T will appear as functions of x, y and z only.

Again, what are known as the equations of the "tetrahedron of stress," which are simply equations (2) applied to the exterior surface of the body, are the following:

$$N_1 \cos p + T_3 \cos q + T_2 \cos r = P \cos \pi\,,$$
$$T_3 \cos p + N_2 \cos q + T_1 \cos r = P \cos \chi\,,$$
$$T_2 \cos p + T_1 \cos q + N_3 \cos r = P \cos \rho\,,$$

in which p, q and r are the angles made with the co-ordinate axes by a normal to the exterior surface at the point where the intensity P of the external force exists, and π, χ and ρ are the angles made by the direction of P with the same axes. Now, if the intensities N and T, as determined by equations (2), are functions of the nature of the material, the intensity of the externally applied force, P, is also dependent, always, on the nature of the material, which is evidently absurd. From these considerations there is deduced the

important principle, that *all problems of elastic equilibrium are completely determinate.*

It is supposed, of course, that the body has assumed its position of equilibrium; this in all ordinary cases is essentially the same as the position of no stress.

It follows immediately from the principle just enunciated that the results of this discussion are applicable to all kinds of material, whether crystalline or not, and under all degrees of stress, even up to the breaking point.

The assumption, at the beginning, of a homogeneous material with deduced results entirely independent of the nature of the material (except for deflections), emphasizes, as has been remarked, the proof of the principle first stated.

The writer regrets exceedingly being so situated that he has no apparatus at his command, otherwise the results of the preceding analysis would have been put to the test of experiment.

Data from one of the many experiments of Kirkaldy will only, therefore, be used in the moment of resistance of a rectangular beam. The bar broken was of Swedish iron two inches square, placed on supports twenty-five inches apart. The weight placed at the centre which broke the bar was 14,000 pounds. The breaking moment of the external forces at the middle section was therefore 87,500 inch pounds. The ultimate tensile resistance of the same iron was found to be about twenty-one tons (2000 pounds per ton). Consequently in equation (74) $N_0 = 21$, $b = 2$ and $z_1 = 2$. These values substituted give

$$M = 61100 \text{ inch-pounds.}$$

By the "common theory" the moment of resistance would have been only about 56000 inch-pounds. Leaving out of consideration the effects of lateral contraction and expansion, therefore, the *apparent* intensity of stress at the point of rupture would be $\frac{87500}{61100} \times 21 = 30$ tons or 60000 pounds.

It is seen from the preceding example that there is a wide discrepancy between the result of experiment and of the formulæ; of which more will be said farther on.

Figure 7 gives the results of the example graphically. Tan β is the tangent of the inclination of the curve to a vertical line at the extremity of the ordinate N. In general, as has already been shown, $\tan \beta = \frac{N_0}{\log z} \frac{1}{z}$. The depth of the beam is two inches.

$z = 1$ inch	$N = 0$	$\tan\beta = 30.3$	$\beta = 88°\ 7'$
$z = 1.25$ inches	$N = 6.76$	$\tan\beta = 24.2$	$\beta = 87°\ 38'$
$z = 1.50$ inches	$N = 12.29$	$\tan\beta = 20.2$	$\beta = 87°\ 10'$
$z = 1.75$ inches	$N = 16.96$	$\tan\beta = 17.3$	$\beta = 86°\ 41'$
$z = 2.00$ inches	$N = 21.00$	$\tan\beta = 15.15$	$\beta = 86°\ 13'$.

The scale of the figure is full size for z and for N, one twentieth of an inch for each ton, or twenty tons for each inch.

The values for β suppose one ton to the inch. They serve to show the varying inclination of the curve, but of course are not found in the figure.

The straight and dotted line shows the law of the "common theory" for the same beam, and illustrates what has been said before, that it is a moderately close approximation to the actual state of stress in a bent beam.

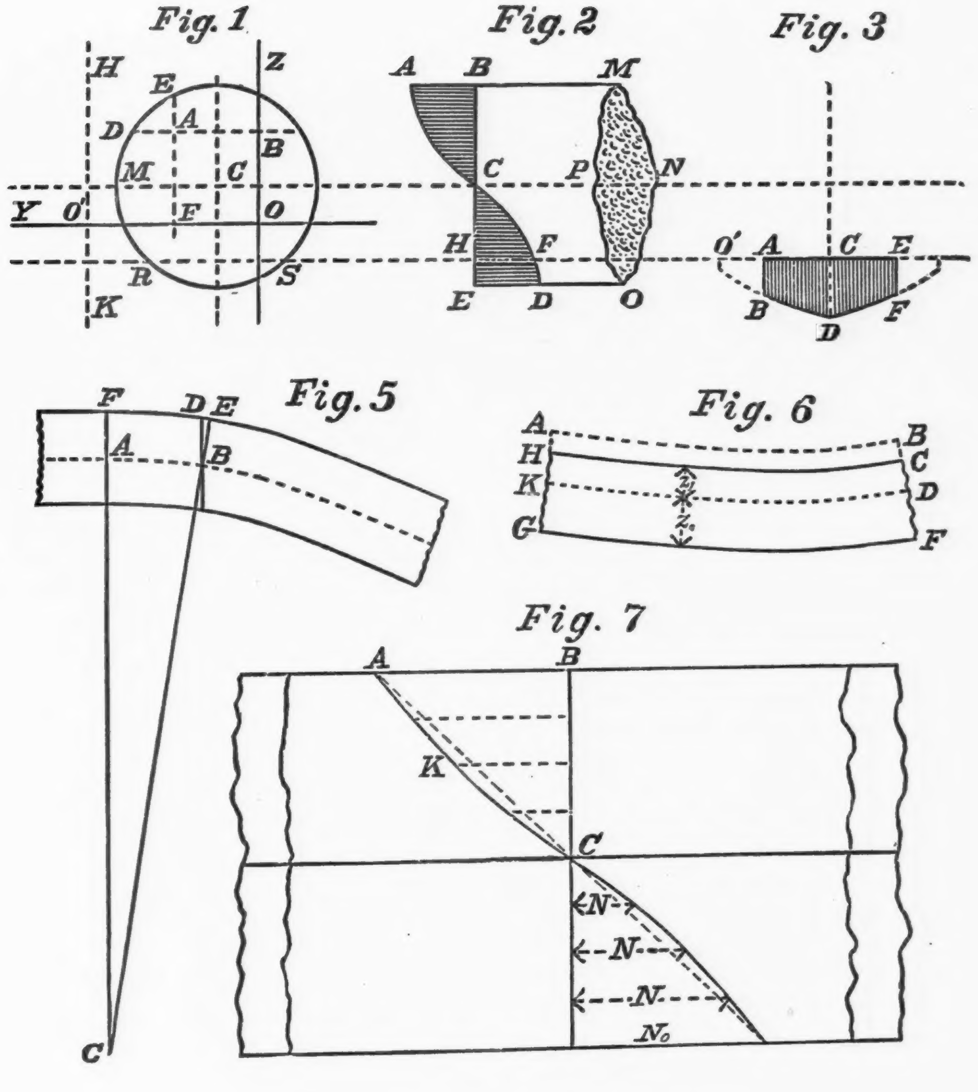
In regard to the discrepancy between the value of $M = 61100$ inch-pounds and the actual value determined by experiment, 87500 inch-pounds, much may be written; but the only way by which an explanation can possibly be arrived at, is that of experiment.

In the first place equation (74) could not possibly give a result coincident with that given by Kirkaldy because in it the effect of the lateral distortion of the fibres on the value of N_0 is neglected. The support which the fibres give each other in resisting lateral contraction or expansion is believed by the writer to be the sole cause of the discrepancy between the result of the formula and that of experiment. This support could not be given were the fibres strained uniformly; in flexure, however, only those fibres equi-distant from the neutral surface are strained the same. It is known that the ultimate resistance of a bar of iron in tension is very much increased if, by any means, lateral contraction can be prevented, and the same is evidently true for compression.

The exact effect of retaining the original area of cross section can only be determined by the aid of experiments, and the writer believes that this branch of the resistance of materials offers a most fruitful and important field of experimental research, of which the limits have yet scarcely been passed.

The curve showing the intensity of stress at any point in the actual case will then probably be found to be that given in Fig. 6, but the co-ordinates representing N will have a considerable increase in length.

Plate I.



BURR, *On the Theory of Flexure.*

To illustrate the effect of resisting the lateral distortion of the fibres the following procedure may be employed. In equation (1), as is sometimes done, suppose $\frac{dv}{dy} = \frac{dw}{dz} = \frac{1}{4}\frac{du}{dx}$ and $\lambda = 2\mu$; then there results $N_1 = 3\mu\frac{du}{dx}$. If there is no lateral contraction then $\frac{dv}{dy} = \frac{dw}{dz} = 0$ and $N_1 = 4\mu\frac{du}{dx}$, giving an increase of $\frac{1}{3}$ over the result obtained with lateral contraction.

It is not by any means an insignificant fact that the same increase in the example taken would almost entirely make up the discrepancy observed.

Now in regard to the method by which N was established in equation (32) and those following. The principles there applied are perfectly general not being restricted to any assumptions or kind of material; they may be applied in absolutely all cases.

The restriction in the application lies in making N a function of z and y *only*, for any given section, and in the present case, as has been shown, that does not affect the generality of the results.

It is believed that the principle of least resistance has not heretofore been applied in the discussion of this problem.

It is also believed that the determinateness of the problems of elastic equilibrium has not before been so generally stated. Clebsch in his admirable work on the theory of elasticity gives a demonstration of the principle, which, however, appears to the writer to be somewhat unsatisfactory.

Note on the First English Euclid.

By George Bruce Halsted, *Tutor in Princeton College, late Fellow of Johns Hopkins University.*

Some interesting questions may now be answered authoritatively, since it is discovered that Princeton possesses, and has possessed for nearly a century, perhaps longer, the identical volume from which the first translation of Euclid into English was made three hundred years ago by Sir Henry Billingsley.

The first translation of Euclid into Latin was made from the Arabic by Adelard of Bath (1130). It is related that he travelled in the East and Spain, where he obtained MSS. From the fact that this version was spread abroad on the Continent with a commentary by Campanus of Novara, it soon began to be attributed to Campanus. It was published at Venice in 1482, and was the first *printed* edition of Euclid. From this or its reprints (1491 and 1516) it has always been taught that the first version into our language was made; see for example the Introduction to Pott's Euclid, Cambridge, 1845, which states, "to Henry Billingsley, a citizen of London, is due the merit of making the first English translation of Euclid's Elements of Geometry. It was made chiefly *from the Latin of Campanus*, and was published in 1570."

There was some dispute as to the extent to which Greek was studied in England at that period, but De Morgan, by a comparison of the Greek of Gregory's Edition with the Latin of Adelard-Campanus and the English of Billingsley, arrived at the belief, in 1837, that this English translation was either made from the Greek or corrected by the Greek.

As the preface was written by the celebrated Dr. John Dee, De Morgan supposed that perhaps he might have furnished the requisite knowledge of Greek.

There seems to be a tendency to doubt Sir Henry Billingsley's erudition, for no reason that I can discover except that he was wealthy and became Lord Mayor of London in 1591.

But now for the new facts. The large folio volume above referred to, in the Library at Princeton, contains first a copy of the first printed edition of Euclid's Elements in Greek, published at Basle in 1533 by John Hervagius, edited by Simon Grynaeus. The text is that of Theon's Revision, and was for a century and three-quarters the only printed Greek text of all the books. Theon was the President of the Neo-Platonic School at Alexandria at the close of the 4th Century. He was the father of the celebrated Hypatia, who succeeded him in the Presidency, and who was assassinated by the Christians in 415.

Appended to this is a copy of the Commentary of Proclus on the First Book of Euclid, printed also at the press of Hervagius in 1533. The editor mentioned, Simon Grynaeus, is the man accused by Anthony Wood of stealing rare MSS. from Oxford. Says Wood, . . . "he took some away, and conveyed them with him beyond the seas, as in an epistle by him written to John, son of Thomas More, he confesseth."

Bound together with these works in Greek, the volume also contains the two-fold Latin translation printed at Basle by Hervagius in 1558. One is the Adelard-Campanus version, from the Arabic; the other is the first translation into Latin from the Greek, made by Zamberti from a MS. of Theon's Revision, and first published at Venice in 1505, twenty-eight years before the appearance of the Editio princeps in Greek.

At the head of this second part of the volume is an address to the reader by Philip Melancthon, dated "Wittenbergæ, mense Augusto, M. D. XXXVII."

Now, all this forms a collection exceedingly rare and valuable in itself; but what gives to this volume its special archæological interest is the fact that it belonged to Billingsley, and was his equipment for the first English Euclid. On the title-page is the autograph signature "*Henricus Billingsley,*" in a most beautiful antique hand. Throughout the volume are very numerous corrections, additions and marginal notes, all in Billingsley's peculiar and beautiful writing. I dare hazard that no Lord Mayor, since his time, has ever written so charming a hand. By reading what he has done, it immediately appears that though he had the Adelard-Campanus Latin before him, yet he gave his special work to a careful comparison of Zamberti's Translation with the original Greek, and the corrections he has actually made sufficiently prove his scholarship and render entirely unnecessary De Morgan's suppositious aid

from Dr. Dee, while, on the other hand, they establish the conclusion about the translation to which De Morgan's sagacity had led him, that "It was certainly made from the Greek, and not from any of the Arabico-Latin versions."

To the one sentence of comparison in proof of this published by De Morgan, Billingsley's autograph indications would enable me to add as many as any one desired, but suffice it to say, that the definitions of the Eleventh Book are alone entirely decisive of the matter.

PRINCETON, *January 9, 1879.*

On the Fundamental Formulae of Dynamics.

By J. W. Gibbs, *New Haven, Conn.*

Formation of a new Indeterminate Formula of Motion by the Substitution of the Variations of the Components of Acceleration for the Variations of the Coordinates in the usual Formula.

The laws of motion are frequently expressed by an equation of the form

$$\Sigma\,[(X - m\ddot{x})\,\delta x + (Y - m\ddot{y})\,\delta y + (Z - m\ddot{z})\,\delta z] = 0, \tag{1}$$

in which

m denotes the mass of a particle of the system considered,
x, y, z its rectangular coordinates,
$\ddot{x}, \ddot{y}, \ddot{z}$ the second differential coefficients of the coordinates with respect to the time,
X, Y, Z the components of the forces acting on the particle,
$\delta x, \delta y, \delta z$ any arbitrary variations of the coordinates which are simultaneously possible, and
Σ a summation with respect to all the particles of the system.

It is evident that we may substitute for $\delta x, \delta y, \delta z$ any other expressions which are capable of the same and only of the same sets of simultaneous values.

Now if the nature of the system is such that certain functions A, B, etc. of the coordinates must be constant, or given functions of the time, we have

$$(2)\quad \left\{ \begin{array}{l} \Sigma\left(\frac{dA}{dx}\,\delta x + \frac{dA}{dy}\,\delta y + \frac{dA}{dz}\,\delta z\right) = 0, \\ \Sigma\left(\frac{dB}{dx}\,\delta x + \frac{dB}{dy}\,\delta y + \frac{dB}{dz}\,\delta z\right) = 0, \\ \text{etc.} \end{array} \right.$$

These are the *equations of condition*, to which the variations in the general equation of motion (1) are subject. But if A is constant or a determined function of the time, the same must be true of $\dot{A}$ and $\ddot{A}$. Now

$$\dot{A} = \Sigma\left(\frac{dA}{dx}\,\dot{x} + \frac{dA}{dy}\,\dot{y} + \frac{dA}{dz}\,\dot{z}\right)$$

and

$$\ddot{A} = \Sigma \left(\frac{dA}{dx}\,\ddot{x} + \frac{dA}{dy}\,\ddot{y} + \frac{dA}{dz}\,\ddot{z} \right) + H,$$

where H represents terms containing only the second differential coefficients of A with respect to the coordinates, and the first differential coefficients of the coordinates with respect to the time. Therefore, if we conceive of a variation affecting the accelerations of the particles at the time considered, but not their positions or velocities, we have

(3) $$\delta\ddot{A} = \Sigma \left(\frac{dA}{dx}\,\delta\ddot{x} + \frac{dA}{dy}\,\delta\ddot{y} + \frac{dA}{dz}\,\delta\ddot{z} \right) = 0,$$

and, in like manner,

$$\delta\ddot{B} = \Sigma \left(\frac{dB}{dx}\,\delta\ddot{x} + \frac{dB}{dy}\,\delta\ddot{y} + \frac{dB}{dz}\,\delta\ddot{z} \right) = 0,$$

etc.

Comparing these equations with (2), we see that when the *accelerations* of the particles are regarded as subject to the variation denoted by δ, but not their positions or velocities, the possible values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ are subject to precisely the same restrictions as the values of δx, δy, δz, when the *positions* of the particles are regarded as variable. We may, therefore, write for the general equation of motion

(4) $$\Sigma \left[(X - m\ddot{x})\,\delta\ddot{x} + (Y - m\ddot{y})\,\delta\ddot{y} + (Z - m\ddot{z})\,\delta\ddot{z} \right] = 0,$$

regarding the positions and velocities of the particles as unaffected by the variation denoted by δ,—a condition which may be expressed by the equations

(5) $$\begin{cases} \delta x = 0, \quad \delta y = 0, \quad \delta z = 0, \\ \delta\dot{x} = 0, \quad \delta\dot{y} = 0, \quad \delta\dot{z} = 0. \end{cases}$$

We have so far supposed that the conditions which restrict the possible motions of the systems may be expressed by *equations* between the coordinates alone or the coordinates and the time. To extend the formula of motion to cases in which the conditions are expressed by the characters $\leqq$ or $\geqq$, we may write

(6) $$\Sigma \left[(X - m\ddot{x})\,\delta\ddot{x} + (Y - m\ddot{y})\,\delta\ddot{y} + (Z - m\ddot{z})\,\delta\ddot{z} \right] \leqq 0.$$

The conditions which determine the possible values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ will not, in such cases, be entirely similar to those which determine the possible values of δx, δy, δz, when the coordinates are regarded as variable. Nevertheless,

the laws of motion are correctly expressed by the formula (6), while the formula

$$(7) \qquad \Sigma \left[(X - m\ddot{x})\,\delta x + (Y - m\ddot{y})\,\delta y + (Z - m\ddot{z})\,\delta z \right] \leqq 0\,,$$

does not, as naturally interpreted, give so complete and accurate an expression of the laws of motion.

This may be illustrated by a simple example.

Let it be required to find the acceleration of a material point, which, at a given instant, is moving with given velocity on the frictionless surface of a body (which it cannot penetrate, but which it may leave), and is acted on by given forces. For simplicity, we may suppose that the normal to the surface, drawn outward from the moving point at the moment considered, is parallel to the axis of X and in the positive direction. The only restriction on the values of δx, δy, δz is that

$$\delta x \geqq 0\,.$$

Formula (7) will therefore give

$$\ddot{x} \geqq \frac{X}{m}, \qquad \ddot{y} = \frac{Y}{m}, \qquad \ddot{z} = \frac{Z}{m}.$$

The condition that the point shall not penetrate the body gives another condition for the value of $\ddot{x}$. If the point remains upon the surface, $\ddot{x}$ must have a certain value N, determined by the form of the surface and the velocity of the point. If the value of $\ddot{x}$ is less that this, the point must penetrate the body. Therefore,

$$\ddot{x} \geqq N\,.$$

But this does not suffice to determine the acceleration of the point.

Let us now apply formula (6) to the same problem. Since $\ddot{x}$ cannot be less than N,

$$\text{if } \ddot{x} = N, \quad \delta\ddot{x} \geqq 0\,.$$

This is the only restriction on the value of $\delta\ddot{x}$, for if $\ddot{x} > N$, the value of $\delta\ddot{x}$ is entirely arbitrary. Formula (6), therefore, requires that

$$\text{if } \ddot{x} = N, \quad \ddot{x} \geqq \frac{X}{m}\,;$$

$$\text{but if } \ddot{x} > N, \quad \ddot{x} = \frac{X}{m}:$$

—that is, (since $\ddot{x}$ cannot be less than N), that $\ddot{x}$ shall be equal to the greater of the quantities N and $\frac{X}{m}$, or to both, if they are equal,—and that

$$\ddot{y} = \frac{Y}{m}, \qquad \ddot{z} = \frac{Z}{m}.$$

The values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are therefore entirely determined by this formula in connection with the conditions afforded by the constraints of the system.*

The following considerations will show that what is true in this case is also true in general, when the conditions to which the system is subject are such that certain functions of the coordinates cannot exceed certain limits, either constant or variable with the time. If certain values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ (with unvaried values of x, y, z, and $\dot{x}$, $\dot{y}$, $\dot{z}$) are simultaneously possible at a given instant, equal or proportional values with the same signs, must be possible for δx, δy, δz immediately after the instant considered, and must satisfy formula (1), and therefore (6), in connection with the values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, X, Y, Z immediately after that instant. The values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, thus determined, are of course the very quantities which we wish to obtain, since the acceleration of a point at a given instant does not denote anything different from its acceleration immediately after that instant.

For an example of a somewhat different class of cases, we may suppose that in a system, otherwise free, x cannot have a negative value. Such a condition does not seem to affect the possible values of δx, as naturally interpreted in a dynamical problem. Yet, if we should regard the value of δx in (7) as arbitrary, we should obtain

$$\ddot{x} = \frac{X}{m},$$

which might be erroneous. But if we regard δx as expressing a velocity of which the system, if at rest, would be capable, (which is not a natural signification of the expression,) we should have $\delta x \geqq 0$, which, with (7), gives

$$\ddot{x} \geqq \frac{X}{m}.$$

This is not incorrect, but it leaves the acceleration undetermined. If we should regard δx as denoting such a variation of the velocity as is possible for the system when it has its given velocity (this also is not a natural

* The failure of the formula (7) in this case is rather apparent than real; for, although the formula apparently allows to $\ddot{x}$, at the instant considered, a value exceeding both N and $\frac{X}{m}$, it does not allow this for any interval, however short. For if $\ddot{x} < N$, the point will immediately leave the surface, and then the formula requires that $\ddot{x} = \frac{X}{m}$.

signification of the expression), formula (7) would give the correct value of $\ddot{x}$ except when $\dot{x} = 0$. In this case (which cannot be regarded as exceptional in a problem of this kind), we should have $\delta x \geqq 0$, which will leave $\ddot{x}$ undetermined, as before.

The application of formula (6), in problems of this kind, presents no difficulty. From the condition

$$\dot{x} \geqq 0,$$

we obtain, first, if $\dot{x} = 0$, $\ddot{x} \geqq 0$,

then, if $\dot{x} = 0$ and $\ddot{x} = 0$, $\delta\ddot{x} \geqq 0$,

which is the only limitation on the value of $\delta\ddot{x}$. With this condition, we deduce from (6) that either

$$\dot{x} = 0, \quad \ddot{x} = 0, \quad \text{and} \quad \ddot{x} \geqq \frac{X}{m};$$

or

$$\ddot{x} = \frac{X}{m}.$$

That is, if $\dot{x} = 0$, $\ddot{x}$ has the greater of the values $\frac{X}{m}$ and 0; otherwise, $\ddot{x} = \frac{X}{m}$.

In cases of this kind also, in which the function which cannot exceed a certain value involves the velocities (with or without the coordinates), one may easily convince himself that formula (6) is always valid, and always sufficient to determine the accelerations with the aid of the conditions afforded by the constraints of the system.

But instead of examining such cases in detail, we shall proceed to consider the subject from a more general point of view.

Comparison of the New Formula with the Statical Principle of Virtual Velocities.—Case of Discontinuous Changes of Velocity.

Formula (1) has so far served as a point of departure. The general validity of this, the received form of the indeterminate equation of motion, being assumed, it has been shown that formula (6) will be valid and sufficient, even in cases in which both (1) and (7) fail. We now proceed to show that the statical principle of *virtual velocities*, when its real signification is carefully considered, leads directly to formula (6), or to an analogous formula for the determination of the discontinuous changes of velocity, when such

occur. This will be the case even if we start with the usual analytical expression of the principle

$$\text{(8)} \qquad \Sigma\,(X\delta x + Y\delta y + Z\delta z) \leqq 0,$$

to which, at first sight, formula (6) appears less closely related than (7). For the variations of the coordinates in this formula must be regarded as relating to differences between the configuration which the system has at a certain time, and which it will continue to have in case of equilibrium, and some other configuration which the system might be supposed to have at some subsequent time. These temporal relations are not indicated explicitly in the notation, and should not be, since the statical problem does not involve the time in any quantitative manner. But in a dynamical problem, in which we take account of the time, it is hardly natural to use δx, δy, δz in the same sense. In any problem in which x, y, z are regarded as functions of the time, δx, δy, δz are naturally understood to relate to differences between the configuration which the system has at a certain time, and some other configuration which it might (conceivably) have had at that time *instead of* that which it actually had.

Now when we suppose a point to have a certain position, specified by x, y, z, at a certain time, its position at that time is no longer a subject of hypothesis or of question. It is its future positions which form the subject of inquiry. Its position in the immediate future is naturally specified by

$$x + \dot{x}dt + \frac{1}{2}\ddot{x}dt^2 + \text{etc.}, \quad y + \dot{y}dt + \frac{1}{2}\ddot{y}dt^2 + \text{etc.}, \quad z + \dot{z}dt + \frac{1}{2}\ddot{z}dt^2 + \text{etc.},$$

and we may regard the variations of these expressions as corresponding to the δx, δy, δz of the statical problem. It is evidently sufficient to take account of the first term of these expressions of which the variation is not zero. Now, x, y, z, as has already been said, are to be regarded as constant. With respect to the terms containing $\dot{x}$, $\dot{y}$, $\dot{z}$, two cases are to be distinguished, according as there is, or is not, a finite change of velocity at the instant considered.

Let us first consider the most important case, in which there is no discontinuous change of velocity. In this case, $\dot{x}$, $\dot{y}$, $\dot{z}$ are not to be regarded as variable (by δ), and the variations of the above expressions are represented by

$$\frac{1}{2}\,\delta\ddot{x}\,dt^2, \quad \frac{1}{2}\,\delta\ddot{y}\,dt^2, \quad \frac{1}{2}\,\delta\ddot{z}\,dt^2,$$

which are, therefore, to be substituted for δx, δy, δz in the general formula of equilibrium (8) to adapt it to the conditions of a dynamical problem. By this substitution (in which the common factor $\frac{1}{2}\,dt^2$ may of course be omitted), and the addition of the terms expressing the reaction against acceleration, we obtain formula (6).

But if the circumstances are such that there is (or may be) a discontinuity in the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ at the instant considered, it is necessary to distinguish the values of these expressions before and after the abrupt change. For this purpose, we may apply $\dot{x}$, $\dot{y}$, $\dot{z}$ to the original values, and denote the changed values by $\dot{x} + \Delta\dot{x}$, $\dot{y} + \Delta\dot{y}$, $\dot{z} + \Delta\dot{z}$. The value of x at a time very shortly subsequent to the instant considered, will be expressed by $x + (\dot{x} + \Delta\dot{x})\,dt +$ etc., in which we may regard $\Delta\dot{x}$ as subject to the variation denoted by δ. The variation of the expression is therefore $\delta\Delta\dot{x}\,dt$. Instead of $-m\ddot{x}$, which expresses the reaction against acceleration, we need in the present case $-\Delta\dot{x}$ to express the reaction against the abrupt change of velocity. A reaction against such a change of velocity is, of course, to be regarded as infinite in intensity in comparison with reactions due to acceleration, and ordinary forces (such as cause acceleration) may be neglected in comparison. If, however, we conceive of the system as acted on by impulsive forces, (*i. e.* such as have no finite duration, but are capable of producing finite changes of velocity, and are measured numerically by the discontinuities of velocity which they produce in the unit of mass,) these forces should be combined with the reactions due to the discontinuities of velocity in the general formula which determines these discontinuities. If the impulsive forces are specified by X, Y, Z, the formula will be

$$\text{(9)} \qquad \Big[(\mathrm{X} - m\Delta\dot{x})\,\delta\Delta\dot{x} + (\mathrm{Y} - m\Delta\dot{y})\,\delta\Delta\dot{y} + (\mathrm{Z} - m\Delta\dot{z})\,\delta\Delta\dot{z}\Big] \leqq 0.$$

The reader will remark the strict analogy between this formula and (6), which would perhaps be more clearly exhibited if we should write $\frac{d\dot{x}}{dt}$, $\frac{d\dot{y}}{dt}$, $\frac{d\dot{z}}{dt}$ for $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ in that formula.

But these formulae may be established in a much more direct manner. For the formula (8), although for many purposes the most convenient expression of the principle of virtual velocities, is by no means the most convenient for our present purpose. As the usual name of the principle implies, it holds

true of velocities as well as of displacements, and is perhaps more simple and more evident when thus applied.*

If we wish to apply the principle, thus understood, to a moving system so as to determine whether certain changes of velocity specified by $\Delta\dot{x}$, $\Delta\dot{y}$, $\Delta\dot{z}$ are those which the system will really receive at a given instant, the velocities to be multiplied into the forces and reactions in the most simple application of the principle are manifestly such as may be imagined to be compounded with the assumed velocities, and are therefore properly specified by $\delta\Delta\dot{x}$, $\delta\Delta\dot{y}$, $\delta\Delta\dot{z}$. The formula (9) may therefore be regarded as the most direct application of the principle of virtual velocities to discontinuous changes of velocity in a moving system.

In the case of a system in which there are no discontinuous changes of velocity, but which is subject to forces tending to produce accelerations, when we wish to determine whether certain accelerations, specified by $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, are such as the system will really receive, it is evidently necessary to consider whether any possible variation of these accelerations is favored more than it is opposed by the forces and reactions of the system. The formula (7) expresses a criterion of this kind in the most simple and direct manner. If we regard a force as a tendency to increase a quantity expressed by $\ddot{x}$, the product of the force by $\delta\ddot{x}$ is the natural measure of the extent to which this tendency is satisfied by an arbitrary variation of the accelerations. The principle expressed by the formula may not be very accurately designated by the words *virtual velocities*, but it certainly does not differ from the principle of virtual velocities (in the stricter sense of the term), more than this differs from that of virtual displacements,—a difference so slight that the distinction of the names is rarely insisted upon, and that it is often very difficult to tell which

* Even in Statics, the principle of virtual *velocities*, as distinguished from that of virtual *displacements*, has a certain advantage in respect of its evidence. The demonstration of the principle in the first section of the *Mécanique Analytique*, if velocities had been considered instead of displacements, would not have been exposed to an objection, which has been expressed by M. Bertrand in the following words: "On a objecté, avec raison, à cette assertion de Lagrange l'example d'un point pesant en équilibre au sommet le plus élevé d'une courbe; il est évident qu'un déplacement infiniment petit le ferait descendre, et, pourtant, ce déplacement ne se produit pas." (*Mécanique Analytique*, troisième édition, tome 1, page 22, note de M. Bertrand.) The value of z (the height of the point above a horizontal plane) can certainly be diminished by a displacement of the point, but value of $\dot{z}$ is not affected by any velocity given to the point.

The real difficulty in the consideration of displacements is that they are only possible at a time subsequent to that in which the system has the configuration to which the question of equilibrium relates. We may make the interval of time infinitely short, but it will always be difficult, in the establishing of fundamental principles, to treat a conception of this kind (relating to what is possible after an infinitesimal interval of time) with the same rigor as the idea of velocities or accelerations, which, in the cases to which (9) and (6) respectively relate, we may regard as communicated immediately to the system.

form of the principle is especially intended, even when the principle is enunciated or discussed somewhat at length.

But, although the formulae (7) and (9) differ so little from the ordinary formulae, they not only have a marked advantage in respect of precision and accuracy, but also may be more satisfactory to the mind, in that the changes considered (to which δ relates), are not so violently opposed to all the possibilities of the case as are those which are represented by the variations of the coordinates.* Moreover, as we shall see, they naturally lead to various important laws of motion.

Transformation of the New Formula.

Let us now consider some of the transformations of which our general formula (7) is capable. If we separate the terms containing the masses of the particles from those which contain the forces, we have

$$(10) \qquad \Sigma\,(X\delta\ddot{x} + Y\delta\ddot{y} + Z\delta\ddot{z}) - \Sigma\left[\frac{1}{2}\,m\,\delta\,(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)\right] \leqq 0,$$

or, if we write u for the acceleration of a particle,

$$(11) \qquad \Sigma\,(X\delta\ddot{x} + Y\delta\ddot{y} + Z\delta\ddot{z}) - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \leqq 0.$$

If, instead of terms of the form $X\delta x$, or in addition to such terms, equation (1) had contained terms of the form $P\delta p$, in which p denotes any quantity determined by the configuration of the system, it is evident that these would give terms of the form $P\delta\ddot{p}$ in (7), (10) and (11). For the considerations which justified the substitution of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ for δx, δy, δz in the usual formula

* It may have seemed to some readers of the *Mécanique Analytique*—a work of which the unity of method is one of the most striking characteristics, and that to which its universally recognized artistic merit is in great measure due—that the treatment of dynamical problems in that work is not entirely analogous to the treatment of statical problems. The statical question, whether a system will remain in equilibrium in a given configuration, is determined by Lagrange by considering all possible motions of the system and inquiring whether there is any reason why the system should take any one of them A similar method in dynamics would be based upon a comparison of a proposed motion with all other motions of which the system is capable without violating its kinematical conditions. Instead of this, Lagrange virtually reduces the dynamical problem to a statical one, and considers, not the possible variations of the proposed motion, but the motions which would be possible if the system were at rest. This reduction of a given problem to a simpler one, which has already been solved, is a method which has its advantages, but it is not the characteristic method of the *Mécanique Analytique*. That which most distinguishes the plan of this treatise from the usual type is the direct application of the general principle to each particular case.

The point is perhaps of small moment, and may be differently regarded by others, but it is mentioned here because it was a feeling of this kind (whether justified or not) and the desire to express the formula of motion by means of a maximum or minimum condition, in which the conditions under which the maximum or minimum subsists should be such as the problem naturally affords, (Gauss's principle of *least constraint* being at the time unknown to the present writer, and the conditions under which the minimum subsists in the principle of *least action* being such that that is hardly satisfactory as a fundamental principle,) which led to the formulae proposed in this paper.

were in no respect dependent upon the fact that x, y, z denote rectangular coordinates, but would apply equally to any other quantities which are determined by the configuration of the system.

Hence, if the moments of all the forces of the system are represented by the sum
$$\mathfrak{S}\,(Pdp)\,,$$
the general formula of motion may be written

$$(12)\qquad \mathfrak{S}\,(P\delta\ddot{p}) - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \leqq 0\,.$$

If the forces admit of a force-function V, we have

$$\delta\,\ddot{V} - \delta\,\Sigma\left(\frac{1}{2}\,mu^2\right) \leqq 0\,,$$

or

$$(13)\qquad \delta\left[\ddot{V} - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \leqq 0\,.$$

But if the forces are determined in any way whatever by the configuration and velocities of the system, with or without the time, X, Y, Z and P will be unaffected by the variation denoted by δ, and we may write the formula of motion in the form

$$(14)\qquad \delta\Sigma\left(X\ddot{x} + Y\ddot{y} + Z\ddot{z} - \frac{1}{2}\,mu^2\right) \leqq 0\,,$$

or

$$(15)\qquad \delta\left[\mathfrak{S}\,(P\ddot{p}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \leqq 0\,.$$

If the forces are determined by the configuration alone, or the configuration and the time, $\delta\dot{X}=0$, $\delta\dot{Y}=0$, $\delta\dot{Z}=0$, $\delta\dot{P}=0$, and the general formula may be written

$$(16)\qquad \delta\left[\frac{d}{dt}\,\Sigma\,(X\dot{x} + Y\dot{y} + Z\dot{z}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \leqq 0\,,$$

or

$$(17)\qquad \delta\left[\frac{d}{dt}\,\mathfrak{S}\,(P\dot{p}) - \Sigma\left(\frac{1}{2}\,mu^2\right)\right] \leqq 0\,.$$

The quantity affected by δ in any one of the last five formulae has not only a maximum value, but absolutely the greatest value consistent with the constraints of the system. This may be shown in reference to (15) by giving to $\ddot{p}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, contained explicitly or implicitly in the expression affected by δ, any possible finite increments $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and subtracting the original value of the expression from the value thus modified. Now,

$$\mathfrak{S}[P(\ddot{p}+\ddot{p}')] - \Sigma\left[\frac{1}{2}m\{(\ddot{x}+\ddot{x}')^2+(\ddot{y}+\ddot{y}')^2+(\ddot{z}+\ddot{z}')^2\}\right] - \mathfrak{S}(P\ddot{p}) + \Sigma\left[\frac{1}{2}m(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)\right]$$
$$= \mathfrak{S}(P\ddot{p}') - \Sigma\,[m\,(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')] - \Sigma\left[\frac{1}{2}\,m\,(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)\right].$$

But since $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ are proportional to and of the same sign with possible values of $\delta\ddot{p}$, $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$, we have, by the general formula of motion,

$$\mathfrak{S}(P\ddot{p}') - \Sigma\left[m(xx' + yy' + zz')\right] \leqq 0.$$

The second member of the preceding equation is therefore negative. The first member is therefore negative, which proves the proposition with respect to (15). The demonstration is precisely the same with respect to (13) and (14), which may be regarded as particular cases of (15).

To show the same with regard to (16) and (17), we have only to observe that the quantities affected by δ in these formulae differ from those affected by the same symbol in (14) and (15) only by the terms

$$\Sigma(\dot{X}\dot{x} + \dot{Y}\dot{y} + \dot{Z}\dot{z}) \quad \text{and} \quad \mathfrak{S}(\dot{P}\dot{p}),$$

which will not be affected by any change in the accelerations of the system.

When the forces are determined by the configuration (with or without the time), the principle may be enunciated as follows: The accelerations in the system are always such that the acceleration of the rate of work done by the forces diminished by one-half the sum of the products of the masses of the particles by the squares of their accelerations has the greatest possible value.

The formula (17), although in appearance less simple than (15), not only is more easily enunciated in words, but has the advantage that the quantity $\frac{d}{dt}\mathfrak{S}(P\dot{p})$ is entirely determined by the system with its forces and motions, which is not the case with $\mathfrak{S}(P\ddot{p})$. The value of the latter expression depends upon the manner in which we choose to represent the forces. For example, if a material point is revolving in a circle under the influence of a central force, we may write either $X\ddot{x} + Y\ddot{y} + Z\ddot{z}$ or $R\ddot{r}$ for $P\ddot{p}$, R and r denoting respectively the force and radius vector. Now $X\ddot{x} + Y\ddot{y} + Z\ddot{z}$ is manifestly unequal to $R\ddot{r}$. But $X\dot{x} + Y\dot{y} + Z\dot{z}$ is equal to $R\dot{r}$, and $\frac{d}{dt}(X\dot{x} + Y\dot{y} + Z\dot{z})$ is equal to $\frac{d}{dt}(R\dot{r})$.

It may not be without interest to see what shape our general formulae will take in one of the most important cases of forces dependent upon the velocities. If a body which can be treated as a point is moving in a medium which presents a resistance expressed by any function of the velocity, the terms due to that resistance in the general formula of motion may be expressed in the form

$$\delta\left[\phi(v)\frac{\dot{x}}{v}\ddot{x} + \phi(v)\frac{\dot{y}}{v}\ddot{y} + \phi(v)\frac{\dot{z}}{v}\ddot{z}\right],$$

where v denotes the velocity and $\phi\,(v)$ the resistance. But

$$\frac{\dot{x}\ddot{x}}{v}+\frac{\dot{y}\ddot{y}}{v}+\frac{\dot{z}\ddot{z}}{v}=\frac{dv}{dt}=\dot{v}\,.$$

The terms due to the resistance reduce, therefore, to

$$\delta\,[\phi\,(v)\,\dot{v}]\,,$$

or,

$$\delta\,\frac{d}{dt}f\,(v)\,,$$

where f denotes the primitive of the function denoted by ϕ.

Discontinuous Changes of Velocity.—Formula (9), which relates to discontinuous changes of velocity, is capable of similar transformations. If we set

$$w^2=\Delta\dot{x}^2+\Delta\dot{y}^2+\Delta\dot{z}^2,$$

the formula reduces to

$$(18)\qquad \delta\,\Sigma\left(\mathrm{X}\Delta\dot{x}+\mathrm{Y}\Delta\dot{y}+\mathrm{Z}\Delta\dot{z}-\frac{1}{2}\,mw^2\right)\leqq 0\,,$$

where X, Y, Z are to be regarded as constant. If $\mathfrak{S}\,(\mathrm{P}dp)$ represents the sum of the moments of the impulsive forces, and we regard P as constant, we have

$$(19)\qquad \delta\left[\mathfrak{S}\,(\mathrm{P}\Delta\dot{p})-\Sigma\left(\frac{1}{2}\,mw^2\right)\right]\leqq 0\,.$$

The expressions affected by δ in these formulae have a greater value than they would receive from any other changes of velocity consistent with the constraints of the system.

Deduction of other Properties of Motion.

The principles which have been established furnish a convenient point of departure for the demonstration of various properties of motion relating to *maxima* and *minima*. We may obtain several such properties by considering how the accelerations of a system, at a given instant, will be modified by changes of the forces or of the constraints to which the system is subject. Let us suppose that the forces X, Y, Z of a system receive the increments X', Y', Z', in consequence of which, and of certain additional constraints, which do not produce any discontinuity in the velocities, the components of acceleration $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ receive the increments $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$. The expression

$$(20)\quad \Sigma\Big[(X+X')(\ddot{x}+\ddot{x}')(Y+Y')(\ddot{y}+\ddot{y}')+(Z+Z')(\ddot{z}+\ddot{z}')$$
$$-\frac{1}{2}\,m\left\{(\ddot{x}+\ddot{x}')^2+(\ddot{y}+\ddot{y}')^2+(\ddot{z}+\ddot{z}')^2\right\}\Big]$$

will be the greatest possible for any values of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ consistent with the constraints. But this expression may be divided into three parts,

$$(21)\quad \Sigma\left[(X+X')\ddot{x}+(Y+Y')\ddot{y}+(Z+Z')\ddot{z}-\frac{1}{2}m(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)\right],$$

$$(22)\quad \Sigma\,[X\ddot{x}'+Y\ddot{y}'+Z\ddot{z}'-m(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')],$$

and

$$(23)\quad \Sigma\left[X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}'-\frac{1}{2}m(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)\right].$$

The first part is evidently constant with reference to variations of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and may, therefore, be neglected. With respect to the second part, we observe that by the general formula of motion we have

$$\Sigma\,[X\delta\ddot{x}+Y\delta\ddot{y}+Z\delta\ddot{z}-m(\ddot{x}\delta\ddot{x}+\ddot{y}\delta\ddot{y}+\ddot{z}\delta\ddot{z})]=0$$

for all values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ which are possible and reversible before the addition of the new constraints. But values proportional to $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and of the same sign, are evidently consistent with the original constraints, and when the components of acceleration are altered to $\ddot{x}+\ddot{x}'$, $\ddot{y}+\ddot{y}'$, $\ddot{z}+\ddot{z}'$, variations of these quantities proportional to and of the same sign as $-\ddot{x}'$, $-\ddot{y}'$, $-\ddot{z}'$ are evidently consistent with the original constraints. Now, if these latter variations were not possible before the accelerations were modified by the addition of the new forces and constraints, it must be that some constraint was then operative which afterwards ceased to be so. The expression (22) will, therefore, be equal to zero, provided only that all the constraints which were operative before the addition of the new forces and constraints, remain operative afterwards.* With this limitation, therefore, the expression (23) must have the greatest value consistent with the constraints. This principle may be expressed without reference to rectangular coordinates. If we write u' for the relative acceleration due to the additional forces and constraints, we have

$$u'^2=\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2,$$

and expression (23) reduces to

$$(24)\quad \Sigma\left(X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}'-\frac{1}{2}mu'^2\right).$$

If the sum of the moments of the additional forces which are considered is represented by $\mathbf{S}(Q'dq)$, (the q representing quantities determined by the configuration of the system,) we have

$$\Sigma\,(X'\ddot{x}+Y'\ddot{y}+Z'\ddot{z})=\mathbf{S}\,(Q'\ddot{q}).$$

We may distinguish the values of $\frac{d^2q}{dt^2}$ immediately before and immediately

*As an illustration of the significance of this limitation, we may consider the condition afforded by the impenetrability of two bodies in contact. Let us suppose that if subject only to the original forces and constraints they would continue in contact, but that, under the influence of the additional forces and constraints, the contact will cease. The impenetrability of the bodies then ceases to be operative as a constraint. Such cases form an exception to the principle which is to be established. But there are no exceptions when all the original constraints are expressed by *equations*.

after the application of the additional forces and constraints by the expressions $\ddot{q}$, and $\ddot{q} + \ddot{q}'$. With this understanding, we have, by differentiation of the preceding equation,

$$\Sigma\,[\dot{X}'\dot{x} + \dot{Y}'\dot{y} + \dot{Z}'\dot{z} + X'\,(\ddot{x} + \ddot{x}') + Y\,(\ddot{y} + \ddot{y}') + Z'\,(\ddot{z} + \ddot{z}')]$$
$$= \mathfrak{S}\,[\dot{Q}\dot{q} + Q\,(\ddot{q} + \ddot{q}')]\,;$$

whence it appears that $\Sigma\,(X'\ddot{x}' + Y'\ddot{y}' + Z'\ddot{z}')$ differs from $\mathfrak{S}\,(Q\ddot{q}')$ only by quantities which are independent of the relative accelerations due to the additional forces and restraints. It follows that these relative accelerations are such as to make

$$\mathfrak{S}\,(Q\ddot{q}') - \Sigma\left(\frac{1}{2}\,mu'^2\right) \tag{25}$$

a maximum.

It will be observed that the condition which determines these relative accelerations is of precisely the same form as that which determines absolute accelerations.

An important case is that in which new constraints are added but no new forces. The relative accelerations are determined in this case by the condition that $\Sigma\left(\frac{1}{2}\,mu'^2\right)$ is a minimum. In any case of motion, in which finite forces do not act at points, lines or surfaces, we may first calculate the accelerations which would be produced if there were no constraints, and then determine the relative accelerations due to the constraints by the condition that $\Sigma\left(\frac{1}{2}\,mu'^2\right)$ is a minimum. This is Gauss's principle of *least constraint.**

Again, in any case of motion, we may suppose u to denote the acceleration which would be produced by the constraints alone, and u' the relative acceleration produced by the forces; we then have

$$\Sigma\,[m\,(\ddot{x}\ddot{x}' + \ddot{y}\ddot{y}' + \ddot{z}\ddot{z}')] = 0,$$

whence, if we write u'' for the resultant or actual acceleration,

$$\Sigma\left(\frac{1}{2}\,mu^2\right) + \Sigma\left(\frac{1}{2}\,mu'^2\right) = \Sigma\left(\frac{1}{2}\,mu''^2\right).$$

Moreover, differentiating (25), we obtain

$$\mathfrak{S}\,(Q\delta\ddot{q}') - \Sigma\,[m\,(\ddot{x}'\delta\ddot{x}' + \ddot{y}'\delta\ddot{y}' + \ddot{z}'\delta\ddot{z}')] = 0,$$

---

*This principle may be derived very directly from the general formula (6), or *vice versa*, for $\Sigma\left(\frac{1}{2}\,mu'^2\right)$ may be put in the form

$$\Sigma\left[\frac{1}{2}\,m\left\{\left(\ddot{x} - \frac{X}{m}\right)^2 + \left(\ddot{y} - \frac{Y}{m}\right)^2 + \left(\ddot{z} - \frac{Z}{m}\right)^2\right\}\right],$$

the variation of which, with the sign changed, is identical with the first member of (6).

whence, since $\delta \ddot{q}', \delta \ddot{x}', \delta \ddot{y}', \delta \ddot{z}'$ may have values proportional to $\ddot{q}', \ddot{x}', \ddot{y}', \ddot{z}'$,

$$\mathfrak{S}\,(Q\ddot{q}') = 2\Sigma\left(\frac{1}{2}\,mu'^2\right).$$

These relations are similar to those which exist with respect to *vis viva* and impulsive forces.

Particular Equations of Motion.

From the general formula (12), we may easily obtain particular equations which will express the laws of motion in a very general form.

Let $d\omega_1$, $d\omega_2$, etc., be infinitesimals (not necessarily complete differentials) the values of which are independent, and by means of which we can perfectly define any infinitesimal change in the configuration of the system; and let

$$\dot{\omega}_1 = \frac{d\omega_1}{dt}, \quad \dot{\omega}_2 = \frac{d\omega_2}{dt}, \quad \text{etc.},$$

where $d\omega_1$, $d\omega_2$ are to be determined by the change in the configuration in the interval of time dt; and let

$$\ddot{\omega}_1 = \frac{d\dot{\omega}_1}{dt}, \quad \ddot{\omega}_2 = \frac{d\dot{\omega}_2}{dt}, \quad \text{etc.}$$

Also let

$$U = \Sigma\left(\frac{1}{2}\,m\ddot{u}^2\right).$$

It is evident that U can be expressed in terms of $\dot{\omega}_1$, $\dot{\omega}_2$, etc., $\ddot{\omega}_1$, $\ddot{\omega}_2$, etc., and the quantities which express the configuration of the system, and that (since δ is used to denote a variation which does not affect the configuration or the velocities),

$$\delta U = \frac{dU}{d\ddot{\omega}_1}\,\delta\ddot{\omega}_1 + \frac{dU}{d\ddot{\omega}_2}\,\delta\ddot{\omega}_2 + \text{etc.}$$

Moreover, since the quantities p in the general formula are entirely determined by the configuration of the system

$$\dot{p} = \frac{dp}{d\omega_1}\,\dot{\omega}_1 + \frac{dp}{d\omega_2}\,\dot{\omega}_2 + \text{etc.},$$

where $\dfrac{dp}{d\omega_1}$ denotes the ratio of simultaneous values of dp and $d\omega_1$, when $d\omega_2$ etc., are equal to zero, and $\dfrac{dp}{d\omega_2}$, etc., are to be interpreted on the same principle. Multiplying by P, and taking the sum with respect to the several forces, we have

$$\mathfrak{S}\,(P\dot{p}) = \Omega_1\dot{\omega}_1, + \Omega_2\dot{\omega}_2 + \text{etc.},$$

where

$$\Omega_1 = \mathfrak{S}\left(P\,\frac{dp}{d\omega_1}\right), \quad \Omega_2 = \mathfrak{S}\left(P\,\frac{dp}{d\omega_2}\right), \quad \text{etc.}$$

If we differentiate with respect to t, and take the variation denoted by δ, we obtain

$$\mathfrak{S}\,(P\delta\ddot{p}) = \Omega_1\delta\ddot{\omega}_1 + \Omega_2\delta\ddot{\omega}_2 + \text{etc.}$$

The general formula (12) is thus reduced to the form

$$(26)\qquad \left(\Omega_1 - \frac{dU}{d\ddot{\omega}_1}\right)\delta\ddot{\omega}_1 + \left(\Omega_2 - \frac{dU}{d\ddot{\omega}_2}\right)\delta\ddot{\omega}_2 + \text{etc.} \geqq 0.$$

If the forces have a potential V, we may write

$$(27)\qquad \left(\frac{dV}{d\omega_1} - \frac{dU}{d\ddot{\omega}_1}\right)\delta\ddot{\omega}_1 + \left(\frac{dV}{d\omega_2} - \frac{dU}{d\ddot{\omega}_2}\right)\delta\ddot{\omega}_2 + \text{etc.},$$

where $\frac{dV}{d\omega_1}$ denotes the ratio of dV and $d\omega_1$ when $d\omega_2$, etc., have the value zero, and the analogous expressions are to be interpreted on the same principle.

If the variations $\delta\omega_1$, $\delta\omega_2$, etc., are capable both of positive and of negative values, we must have

$$(28)\qquad \frac{dU}{d\ddot{\omega}_1} = \Omega_1, \quad \frac{dU}{d\ddot{\omega}_2} = \Omega_2, \quad \text{etc.},$$

or,

$$(29)\qquad \frac{dU}{d\ddot{\omega}_1} = \frac{dV}{d\omega_1}, \quad \frac{dU}{d\ddot{\omega}_2} = \frac{dV}{d\omega_2}, \quad \text{etc.}$$

To illustrate the use of these equations in a case in which $d\omega_1$, $d\omega_2$, etc., are not exact differentials, we may apply them to the problem of the rotation of a rigid body of which one point is fixed. If $d\omega_1$, $d\omega_2$, $d\omega_3$ denote infinitesimal rotations about the principal axes which pass through the fixed point, Ω_1, Ω_2, Ω_3 will denote the moments of the impressed forces about these axes, and the value of U will be given by the formula

$$\begin{aligned}2U = (a+b+c)(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2)^2 - (\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2)(a\dot{\omega}_1^2 + b\dot{\omega}_2^2 + c\dot{\omega}_3^2) \\ + 2(b-c)\,\dot{\omega}_2\dot{\omega}_3\ddot{\omega}_1 + 2(c-a)\,\dot{\omega}_3\dot{\omega}_1\ddot{\omega}_2 + 2(a-b)\,\dot{\omega}_1\dot{\omega}_2\ddot{\omega}_3 \\ + (b+c)\,\ddot{\omega}_1^2 + (c+a)\,\ddot{\omega}_2^2 + (a+b)\,\ddot{\omega}_3^2,\end{aligned}$$

where a, b, and c are constants, $a+b$, $b+c$, $c+a$ being the *moments of inertia* about the three axes. Hence,

$$\frac{dU}{d\ddot{\omega}_1} = (b-c)\,\dot{\omega}_2\dot{\omega}_3 + (b+c)\,\ddot{\omega}_1, \quad \frac{dU}{d\ddot{\omega}_2} = (c-a)\,\dot{\omega}_3\dot{\omega}_1 + (c+a)\,\ddot{\omega}_2,$$

$$\frac{dU}{d\ddot{\omega}_3} = (a-b)\,\dot{\omega}_1\dot{\omega}_2 + (a+b)\,\ddot{\omega}_3;$$

and the equations of motion are

$$\ddot{\omega}_1 = \frac{(c-b)\,\dot{\omega}_2\dot{\omega}_3 + \Omega_1}{c+b},$$

$$\ddot{\omega}_2 = \frac{(a-c)\,\dot{\omega}_3\dot{\omega}_1 + \Omega_2}{a+c},$$

$$\ddot{\omega}_3 = \frac{(b-a)\,\dot{\omega}_1\dot{\omega}_2 + \Omega_3}{b+a}.$$

Addenda to Bibliography of Hyper-Space and Non-Euclidean Geometry.

BY GEORGE BRUCE HALSTED.

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* The Relation which can be proved to subsist between the Area of a Plane Triangle and the Sum of the Angles, on the Hypothesis that Euclid's 12th Axiom is False. Read before the Canadian Institute, 25th February, 1860. Published in the Canadian Journal of Industry, Science and Art. New Series, vol. V, 1860, pp. 341–356. "I propose to prove in the present paper, that, if Euclid's 12th Axiom be supposed to fail in any case, a relation subsists between the area of a plane triangle and the sum of the angles. Call the area A, and the sum of the angles s; a right angle being taken as the unit of measure. Then $A = k\ (2 - s)$; k being a constant finite quantity, that is, a finite quantity that remains the same for all triangles. This formula may be considered as holding good even when Euclid's 12th Axiom is assumed to be true; only k is, in that case, infinite." This paper was drawn up without the slightest knowledge whatsoever that anything had ever before

been written or spoken on the subject. Thus the name of Prof. G. P. Young must be added to those of Lobatchewsky and Bolyai as an independent discoverer of the possibility of a pseudo-spherical geometry. The proof, which is in the style of Euclid, is thoroughly elementary, even more so perhaps than Bolyai's, and, like his, is applied to but two of the three geometries of surfaces of constant curvature; the assumption of Euclid's Sixth Postulate in the very first proposition, shutting out spherical geometry. Omitting this proposition, the proof is easily extended to pan-geometry. It is worthy of notice that the proof begins with the very proposition on which Legendre attempted, in the twelfth edition of his Éléments de Géométrie, to found a demonstration of the theory of parallels.

68. Tait, P. G.

I. Mentions Hyper-Space in his Address as Pres. of Math. Sect. of Brit. Assoc. at Edinburgh. Br. Asso. Rep., 1871, p. 3.

II. Recent Advances in Physical Science. Second edition. Introduc., pp. 5–6. London, 1876.

III. Review of Zöllner. Nature, March 28, 1878, pp. 420–422.

69. Engel, G.

I. Der Idee des Raumes und der Räume. Berlin, 1868.

70. Liebmann, O.

I. Zur Analysis der Wirklichkeit. Strassburg. 8°. VI, 619 pp. 1876.

71. Geiser.

I. Sopra una quistione geometrica di massimo e suo estensione ad uno spazio di *n* dimensioni. Milano, Inst. Lomb. Rendiconti I, 1868, pp. 778–783.

72. Fleury, H.

I. La Géométrie affranchie du Postulatum d'Euclide. Paris, 1869. A sad paradoxer, worthy of De Morgan's Budget.

73. Land, J. P. N.

I. Kant's Space and Modern Mathematics. Mind. II., No. 5, Jan., 1877, pp. 38–46.

II. Critical Notice of Erdmann's "Die Axiome der Geometrie." Mind. III, 1878, pp. 551–555.

74. HOFFMANN, J. J. G.

I. Das elfte Axiom der Elemente des Euclides. Halle, 1859.

75. DELBOEUF, J.

*I. Prolégomènes philosophiques de la Géométrie et solutions des postulats. Liége, 1860. An attempt to found the theory of parallels on the homogeneity of space; see Preface, pp. vi and vii. Homogeneity, as defined in the preface and as usually understood, is found to be insufficient (see pp. 144 and 145); in fact by homogeneous M. Delboeuf means both homogeneous and homaloid. He is thus really at one with the new geometry. See also *C.*, pp. 75–84.

76. RODWELL, G. F.

*I. On Space of Four Dimensions. Nature, vol. VIII, pp. 8 and 9, 1873. An attempt to realize (α) a condition of life in space of two dimensions, (β) by the addition of the element of diverse motions to our already known space, the condition of life in space of four dimensions.

77. WITTE, H. TH.

*I. Parallentheorie. Wolfenbüttel, 1867.

78. BOUNEL, F.

*I. Sur les Definitions géométriques. Paris, 1871.

79. KRAUSE, A.

*I. Kant und Helmholtz über den Ursprung und die Bedeutung der Raumanschauung und der geometrischen Axiome. Schauenburg, 1878.

80. SCHLEGEL, V.

I. Ueber neuere geometrische Methoden und ihre Verwandtschaft mit der Grassmann'schen Ausdehnungslehre. Zeitsch. für Math. u. Phys., XXIV, pp. 83–95. Leipzig, 1879.

Calculation of the Minimum N. G. F. of the Binary Seventhic.

By Professor Cayley, *Cambridge, England.*

For the binary seventhic $(a, \dots \between x, y)^7$ the number of the asyzygetic covariants $(a, \dots)^\theta(x, y)^\mu$, or say of the degorder $(\theta . \mu)$ is given as the coefficient of $a^\theta x^\mu$ in the function

$$\frac{1-x^{-2}}{1-ax^7 . 1-ax^5 . 1-ax^3 . 1-ax . 1-ax^{-1} . 1-ax^{-3} . 1-ax^{-5} . 1-ax^{-7}}$$

developed in ascending powers of a. See my Ninth Memoir on Quantics, Phil. Trans., t. CLXI (1871), pp. 17–50.

This function is in fact

$$= A\,(x) - \frac{1}{x^2}\,A\left(\frac{1}{x}\right),$$

where, developing in ascending powers of a, the second term $-\dfrac{1}{x^2}\,A\left(\dfrac{1}{x}\right)$ contains only negative powers of x, and it may consequently be disregarded: the number of asyzygetic covariants of the degorder $(\theta . \mu)$ is thus equal to the coefficient of $a^\theta x^\mu$ in the function $A\,(x)$, which function is for this reason called the Numerical Generating Function (N. G. F.) of the binary seventhic; and the function $A\,(x)$ expressed as a fraction in its least terms is said to be the minimum N. G. F.

According to a theorem of Professor Sylvester's (Proc. Royal Soc. t. XXVIII, 1878, pp. 11–13), this minimum N. G. F. is of the form

$$\frac{Z_0 + aZ_1 + a^2Z_2 \dots \quad + a^{36}Z_{36}}{1-ax . 1-ax^3 . 1-ax^5 . 1-ax^7 . 1-a^4 . 1-a^6 . 1-a^8 . 1-a^{10} . 1-a^{12}},$$

where $Z_0, Z_1, \dots Z_{36}$ are rational and integral functions of x of degrees not exceeding 14; and where, as will presently be seen, there is a symmetry in regard to the terms Z_0, Z_{36}; Z_1, Z_{35}; &c., equidistant from the middle term Z_{18}, such that the terms $Z_0, \dots Z_{18}$ being known, the remaining terms $Z_{19}, \dots Z_{36}$ can be at once written down.

Using only the foregoing properties, I obtained for the N. G. F. an expression which I communicated to Professor Sylvester, and which is published, *Comptes Rendus*, t. LXXXVII, 1878, p. 505, but with an erroneous value for the coefficient of a^7 and for that of the corresponding term a^{29}.* The correct value is

* The existence of these errors was pointed out to me by Professor Sylvester in a letter dated 13th November, 1878.

Numerator of Minimum N. G. F. is =

$$
\begin{aligned}
&1\\
&+ a\ (-x - x^3 - x^5)\\
&+ a^2\ (x^2 + x^4 + 2x^6 + x^8 + x^{10})\\
&+ a^3\ (-x^7 - x^9 - x^{11} - x^{13})\\
&+ a^4\ (2x^4 + x^8 + x^{14})\\
&+ a^5\ (x + 2x^3 - x^9 - x^{11})\\
&+ a^6\ (-1 + 2x^2 - x^4 - x^8 - x^{10} + x^{12})\\
&+ a^7\ (4x + x^3 + 3x^5 - x^9 + x^{11})\\
&+ a^8\ (2 - x^2 - 3x^6 - 3x^8 - x^{10} - x^{12})\\
&+ a^9\ (x + 3x^3 + x^5 - x^7 + 2x^9 + 2x^{13})\\
&+ a^{10} (-1 + 4x^2 - x^6 - 2x^8 - 2x^{10} - x^{14})\\
&+ a^{11} (5x + 3x^3 + 2x^5 - x^7 - 2x^9 - x^{11} + x^{13})\\
&+ a^{12} (5 + x^2 - 4x^6 - 6x^8 - 4x^{10} - x^{12} + 2x^{14})\\
&+ a^{13} (x - 4x^5 - 4x^7 - x^9 + x^{11} + 4x^{13})\\
&+ a^{14} (2 + 5x^2 + x^4 + x^6 - 2x^8 + 3x^{12} - x^{14})\\
&+ a^{15} (3x - x^3 - x^5 - 7x^7 - 5x^9 - x^{11} - x^{13})\\
&+ a^{16} (6 + 3x^2 + 3x^4 - 4x^6 - 3x^8 - x^{12} + 5x^{14})\\
&+ a^{17} (-x - 2x^3 - 9x^5 - 8x^7 - 4x^9 - 3x^{11} + 4x^{13})\\
&+ a^{18} (2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14})\\
&+ a^{19} (4x - 3x^3 - 4x^5 - 8x^7 - 9x^9 - 2x^{11} - x^{13})\\
&+ a^{20} (5 - x^2 - 3x^6 - 4x^8 + 3x^{10} + 3x^{12} + 6x^{14})\\
&+ a^{21} (-x - x^3 - 5x^5 - 7x^7 - x^9 - x^{11} + 3x^{13})\\
&+ a^{22} (-1 + 3x^2 - 2x^4 + x^8 + x^{10} + 5x^{12} + 2x^{14})\\
&+ a^{23} (4x + x^3 - x^5 - 4x^7 - 4x^9 + x^{13})\\
&+ a^{24} (2 - x^2 - 4x^4 - 6x^6 - 4x^8 + x^{10} + 5x^{14})\\
&+ a^{25} (x - x^3 - 2x^5 - x^7 + 2x^9 + 3x^{11} + 5x^{13})\\
&+ a^{26} (-1 - 2x^4 - 2x^6 - x^8 + 4x^{10} - x^{14})\\
&+ a^{27} (2x + 2x^5 - x^7 + x^9 + 3x^{11} + x^{13})\\
&+ a^{28} (-x^2 - x^4 - 3x^6 - 3x^8 - x^{12} + 2x^{14})\\
&+ a^{29} (x^3 - x^5 + 3x^9 + x^{11} + 4x^{13})\\
&+ a^{30} (x^2 - x^4 - x^6 - x^{10} + 2x^{12} - x^{14})\\
&+ a^{31} (-x^3 - x^5 + 2x^{11} + x^{13})\\
&+ a^{32} (1 + x^6 + 2x^{10})\\
&+ a^{33} (-x - x^3 - x^5 - x^7)\\
&+ a^{34} (x^4 + x^6 + 2x^8 + x^{10} + x^{12})\\
&+ a^{35} (-x^9 - x^{11} - x^{13})\\
&+ a^{36} \,.\, x^{14}
\end{aligned}
$$

Denominator (as mentioned before) is

$$=1-ax\,.\,1-ax^3\,.\,1-ax^5\,.\,1-ax^7\,.\,1-a^4\,.\,1-a^6\,.\,1-a^8\,.\,1-a^{10}\,.\,1-a^{12}\,.$$

The method of calculation is as follows: write for a moment

$$\phi\,(a,\,x)=\frac{1-x^{-2}}{1-ax^7.\,1-ax^5.\,1-ax^3.\,1-ax\,.\,1-ax^{-1}.\,1-ax^{-3}.\,1-ax^{-5}.\,1-ax^{-7}},$$

then $\phi\,(a,\,x)$ developed in ascending powers of a, and rejecting from the result all negative powers of x, is

$$=\frac{Z_0+aZ_1+\ldots a^{36}Z_{36}\,.}{1-ax\,.\,1-ax^3.\,1-ax^5.\,1-ax^7.\,1-a^4.\,1-a^6.\,1-a^8.\,1-a^{10}.\,1-a^{12}}$$

developed in like manner in ascending powers of a; for the determination of the Z's up to Z_{18} we require only the development of $\phi\,(a,\,x)$ up to a^{18}; and, assuming that each Z is at most of the degree 14 in x, we require the coefficients of the different powers of a in $\phi\,(a,\,x)$ only up to x^{14}: assuming then that $\phi\,(a,\,x)$ developed in ascending powers of a, up to a^{18}, rejecting all negative powers of x, and all positive powers greater than x^{14}, is

$$=X_0+aX_1+\ldots+a^{18}X_{18}\,.$$

We have

$$X_0+aX_1\ldots+a^{18}X_{18}=\frac{Z_0+aZ_1\ldots+a^{18}Z_{18}}{1-ax.\,1-ax^3.\,1-ax^5.\,1-ax^7.\,1-a^4.\,1-a^6.\,1-a^8.\,1-a^{10}.\,1-a^{12}},$$

or say

$$Z_0+aZ_1+\ldots a^{18}Z_{18}=1-a^4.\,1-a^6.\,1-a^8.\,1-a^{10}.\,1-a^{12}.$$
$$1-ax\,.\,1-ax^3.\,1-ax^5.\,1-ax^7.\,(X_0+aX_1\;.\;.\;.\;+a^{18}X_{18});$$

viz: developing here the right hand side as far as a^{18}, but in each term rejecting the powers of x above x^{14}, the coefficients of the several powers $a^0,\,a^1,\ldots a^{18}$ give the required values $Z_0,\,Z_1,\ldots Z_{18}$. We require, therefore, only to know the values of these functions $X_0,\,X_1\ldots X_{18}$.

To make a break in the calculation, it is convenient to write

$$1-ax.1-ax^3.1-ax^5.\,1-ax^7\,(X_0+aX_1\ldots+a^{18}X_{18})=Y_0+aY_1\ldots+a^{18}Y_{18}\,;$$

putting then

$$1-ax\,.\,1-ax^3.\,1-ax^5.\,1-ax^7=1-ap+a^2q-a^3r\,,$$

where (up to x^{14})

$$p=x+x^3+x^5+x^7$$
$$q=x^4+x^6+2x^8+x^{10}+x^{12}$$
$$r=x^9+x^{11}+x^{13}\,,$$

we have

$$Y_0+aY_1+a^2Y_2\ldots+a^{18}Y_{18}=(1-ap+a^2q-a^3r)(X_0+aX_1+a^2X_2\ldots+a^{18}X_{18}),$$

and the values of $Y_0,\,Y_1,\ldots Y_{18}$ then are

$$\begin{array}{rrrrcr}
 & Y_0 & Y_1 & Y_2 & Y_3 \; . \; . \; . \; . & Y_{18} \\
\hline
= & X_0 & X_1 & X_2 & X_3 & X_{18} \\
 & & -pX_0 & -pX_1 & -pX_2 & -pX_{17} \\
 & & & +qX_0 & +qX_1 & +qX_{16} \\
 & & & & -rX_0 & -rX_{15}
\end{array}$$

the values being taken to x^{14} only; and we then have

$$Z_0+aZ_1+a^2Z_2\,..+a^{18}Z_{18}=1-a^4.1-a^6.1-a^8.1-a^{10}.1-a^{12}(Y_0+aY_1\,..+a^{18}Y_{18})$$

viz: the values of $Z_0, Z_1, \ldots Z_{18}$ are

$$\begin{array}{rrrrrrrrrrr}
 & Z_0 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 & Z_9 \\
\hline
= & Y_0 & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 \\
 & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 & -Y_4 & -Y_5 \\
 & & & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 \\
 & & & & & & & & & -Y_0 & -Y_1
\end{array}$$

$$\begin{array}{rrrrrrrrrr}
 & Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\
\hline
= & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} \\
 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} & -Y_{13} & -Y_{14} \\
 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} \\
 & -Y_2 & -Y_3 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} \\
 & & & & & +2Y_0 & +2Y_1 & +2Y_2 & +2Y_3 & +2Y_4 \\
 & & & & & & & +2Y_0 & +2Y_1 & +2Y_2 \\
 & & & & & & & & & +Y_0.
\end{array}$$

The rule of symmetry, before referred to, is that the coefficient Z_{36-p} of a^{36-p} is obtained from the coefficient Z_p of a^p by changing each power x^q into x^{14-q}, the coefficients being unaltered; in particular Z_{18}, the coefficient of a^{18}, must remain unaltered when each power x^q is changed into x^{14-q}; and the verification thus obtained of the value

$$Z_{18}=2+6x^2+x^4+2x^6+2x^8+x^{10}+6x^{12}+2x^{14}$$

is in fact almost a complete verification of the whole work. Some other verifications, which present themselves in the course of the work, will be referred to further on.

We have, therefore, to calculate the coefficients $X_0, X_1, \ldots X_{18}$; the function $\phi(a, x)$ regarded as a function of a is at once decomposed into simple fractions; viz: we have

$$\phi(a, x)=\frac{1-x^{-2}}{1-ax^7.\,1-ax^5.\,1-ax^3.\,1-ax.\,1-ax^{-1}.\,1-ax^{-3}.\,1-ax^{-5}.\,1-ax^{-7}}$$

$$=\frac{x^{54}}{1-x^4.\,1-x^6.\,1-x^8.\,1-x^{10}.\,1-x^{12}.\,1-x^{14}}\;\frac{1}{1-ax^7}$$

$$
\begin{aligned}
&-\frac{x^{40}}{1-x^2\,.\,1-x^4\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}\,.\,1-x^{12}}\;\frac{1}{1-ax^5}\\
&+\frac{x^{28}}{1-x^2\,.\,(1-x^4)^2\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}}\;\frac{1}{1-ax^3}\\
&-\frac{x^{18}}{1-x^2\,.\,(1-x^4)^2\,.\,(1-x^6)^2\,.\,1-x^8}\;\frac{1}{1-ax}\\
&+\frac{x^{10}}{1-x^2\,.\,(1-x^4)^2\,.\,(1-x^6)^2\,.\,1-x^8}\;\frac{1}{1-ax^{-1}}\\
&-\frac{x^4}{1-x^2\,.\,(1-x^4)^2\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}}\;\frac{1}{1-ax^{-3}}\\
&+\frac{1}{1-x^2\,.\,1-x^4\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}\,.\,1-x^{12}}\;\frac{1}{1-ax^{-5}}\\
&-\frac{x^{-2}}{1-x^4\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}\,.\,1-x^{12}\,.\,1-x^{14}}\;\frac{1}{1-ax^{-7}}\,.
\end{aligned}
$$

In order to obtain the expansion of $\phi\,(a,\,x)$ in the assumed form of an expansion in ascending powers of a, we must of course expand the simple fractions $\frac{1}{1-ax^7}$, &c. in ascending powers of a, but it requires a little consideration to see that we must also expand the x-coefficients of these simple fractions *in ascending powers of x.* For instance, as regards the term independent of a, here developing the several coefficients as far as x^{18}, the last five terms give (see *post*)

$$
\begin{array}{rrrrrrrrrrr}
 & & & & & & & & & & -\;x^{18}\\
 & & & & & +\;x^{10} & +\;x^{12} & +\;3x^{14} & +\;5x^{16} & & +\;9x^{18}\\
 & & & -\;x^4 & -\;x^6 & -\;3x^8 & -\;4x^{10} & -\;8x^{12} & -\;11x^{14} & -\;18x^{16} & -\;24x^{18}\\
 & 1 & +\;x^2 & +\;2x^4 & +\;3x^6 & +\;5x^8 & +\;7x^{10} & +\;11x^{12} & +\;14x^{14} & +\;20x^{16} & +\;26x^{18}\\
-\;x^{-2} & . & -\;x^2 & -\;x^4 & -\;2x^6 & -\;2x^8 & -\;4x^{10} & -\;4x^{12} & -\;6x^{14} & -\;7x^{16} & -\;10x^{18}\\
\hline
=-\;x^{-2} & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

viz: the sum is $=1-x^{-2}$ as it should be.*

The expansion is required only as far as x^{14}: the first four terms are therefore to be disregarded, and, writing for shortness

$$
\begin{aligned}
E &= \frac{1}{1-x^2\,.\,(1-x^4)^2\,(1-x^6)^2\,.\,1-x^8}\\
F &= \frac{1}{1-x^2\,.\,(1-x^4)^2\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}}\\
G &= \frac{1}{1-x^2\,.\,1-x^4\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}\,.\,1-x^{12}}\\
H &= \frac{1}{1-x^4\,.\,1-x^6\,.\,1-x^8\,.\,1-x^{10}\,.\,1-x^{12}\,.\,1-x^{14}}\,,
\end{aligned}
$$

we have $\quad \phi\,(a,\,x) = \dfrac{x^{10}E}{1-ax^{-1}} - \dfrac{x^4F}{1-ax^{-3}} + \dfrac{G}{1-ax^{-5}} - \dfrac{x^{-2}H}{1-ax^{-7}}\,,$

*To give the last degree of perfection to the beautiful method of Professor Cayley it would seem desirable that a proof should be given of the principle illustrated by the example in the text, and the nature of the mischief resulting from its neglect clearly pointed out.—EDS.

which is

$$\begin{aligned}
= &\quad x^{10} E\,(1 + ax^{-1} + a^2x^{-2} + \ldots) \\
- &\quad x^4 F\,(1 + ax^{-3} + a^2x^{-6} + \ldots) \\
+ &\quad G\,(1 + ax^{-5} + a^2x^{-10} + \ldots) \\
- &\,x^{-2} H\,(1 + ax^{-7} + a^2x^{-14} + \ldots)\,,
\end{aligned}$$

where the several series are to be continued up to a^{18}, and, after substituting for E, F, G, H their expansions in ascending powers of x, we are to reject negative powers of x, and also powers higher than x^{14}. The functions E, F, G, H contain each of them only even powers of x, and it is easy to see that we require the expansions up to x^{22}, x^{64}, x^{104} and x^{142} respectively. For the sake of a verification, I in fact calculated E, F up to x^{64} and G, H up to x^{142}, viz: we have $(1-x^6)\,E = (1-x^{10})\,F$, from the coefficients of E we have those of $(1-x^6)\,E$, and in the process of calculating F we have at the last step but one the coefficients of $(1-x^{10})\,F$, the agreement of the two sets being the verification; similarly, $(1-x^2)\,G = (1-x^{14})\,H$ gives a verification. The process for the calculation of $E, = \dfrac{1}{1-x\,.\,(1-x^4)^2\,(1-x^6)^2\,.\,1-x^8}$, is as follows:

Ind. x	0	2	4	6	8	10	12	14	16	18	20	22
$(1-x^2)^{-1}$	1	1	1	1	1	1	1	1	1	1	1	1
			1	1	2	2	3	3	4	4	5	5
$(1-x^4)^{-1}$	1	1	2	2	3	3	4	4	5	5	6	6
			1	1	3	3	6	6	10	10	15	15
$(1-x^4)^{-1}$	1	1	3	3	6	6	10	10	15	15	21	21
				1	1	3	4	7	9	14	17	24
$(1-x^6)^{-1}$	1	1	3	4	7	9	14	17	24	29	38	45
				1	1	3	5	8	12	19	25	36
$(1-x^6)^{-1}$	1	1	3	5	8	12	19	25	36	48	63	81
					1	1	3	5	9	13	22	30
$E = (1-x^8)^{-1}$	1	1	3	5	9	13	22	30	45	61	85	111

the alternate lines giving the developments of the functions $(1-x^2)^{-1}$, $(1-x^2)^{-1}(1-x^4)^{-1}$, $(1-x^2)^{-1}(1-x^4)^{-2}$, ..., which are the products of the x-functions down to any particular line. And in like manner we have the expansions of the other functions F, G, H respectively. I give first the expansions of E, F, G, H; next the calculation of the X's; then the calculation of the Y's: and from these the Z's up to Z_{18}, or coefficients of the powers $a^0, a^1, \ldots a^{18}$ in the numerator of the N. G. F. are at once found; and the coefficients of the remaining powers $a^{19}, \ldots a^{36}$ are then deduced, as already mentioned.

Writing in the formula $x=0$, we have, for the numerator of the N. G. F. of the *invariants*, the expression

$$1-a^6+2a^8-a^{10}+5a^{12}+2a^{14}+6a^{16}+2a^{18}+5a^{20}-a^{22}+2a^{24}-a^{26}+a^{32},$$

agreeing with a result in my second Memoir on Quantics, Phil. Trans., t. CXLVI, (1856), p. 117; this, then, was a known result, and it affords a verification, not only of the terms in x^0, but also of those in x^{14}. Thus, in calculating the foregoing expression of the numerator, we obtain $Z_4=(2x^4+x^8+x^{14})$, viz: the term is $a^4(2x^4+x^8+x^{14})$, and we thence have the corresponding term $a^{32}(1+x^6+2x^{10})$, which, when $x=0$, becomes $=a^{32}$, a term of the numerator for the invariants: and the term $1x^{14}$ of Z_4 is thus verified, viz: so soon as, in the calculation, we arrive at this term, we know that it is right, and the calculation up to this point is, to a considerable extent, verified. And similarly, in continuing the calculation, we arrive at other terms which are verified in the like manner.

EXPANSIONS OF THE FUNCTIONS E, F, G, H.

Ind. x	E	F	G	H	Ind. x	E	F	G	H
0	1	1	1	1	16	45	36	20	6
2	1	1	1	0	18	61	47	26	7
4	3	3	2	1	20	85	66	35	10
6	5	4	3	1	22	111	84	44	11
8	9	8	5	2	24		113	58	16
10	13	11	7	2	26		141	71	17
12	22	18	11	4	28		183	90	23
14	30	24	14	4	30		225	110	26

Ind. x	F	G	H	Ind. x	G	H	Ind. x.	H
32	284	136	33	70	2172	419	108	2265
34	344	163	37	72	2432	472	110	2426
36	425	199	47	74	2702	515	112	2623
38	508	235	52	76	3009	576	114	2807
40	617	282	64	78	3331	629	116	3026
42	729	331	72	80	3692	699	118	3232
44	872	391	86	82	4070	760	120	3479
46	1020	454	96	84	4494	843	122	3708
48	1205	532	115	86	4935	913	124	3981
50	1397	612	127	88	5427	1007	126	4240
52	1632	709	149	90	5942	1091	128	4541
54	1877	811	166	92	6510	1197	130	4828
56	2172	931	192	94	7104	1293	132	5164
58	2480	1057	212	96	7760	1416	134	5481
60	2846	1206	245	98	8442	1525	136	5850
62	3228	1360	269	100	9192	1663	138	6204
64	3677	1540	307	102	9975	1790	140	6609
66		1729	338	104	10829	1945	142	6998
68		1945	382	106		2088		

CALCULATION OF THE X'S.

Ind. x even or odd according as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
						1	1	3
			−1	−1	−3	−4	−8	−11
	1	1	2	3	5	7	11	14
		−1	−1	−2	−2	−4	−4	−6
$X_0 =$	1	0	0	0	0	0	0	0
					1	1	3	
	−1	−1	−3	−4	−8	−11	−18	
	3	5	7	11	14	20	26	
	−2	−4	−4	−6	−7	−10	−11	
$X_1 =$	0	0	0	+1	0	0	0	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
					1	1	3	5
	−1	−3	−4	−8	−11	−18	−24	−36
	7	11	14	20	26	35	44	58
	−6	−7	−10	−11	−16	−17	−23	−26
$X_2 =$	0	+1	0	+1	0	+1	0	+1
				1	1	3	5	
	−4	−8	−11	−18	−24	−36	−47	
	20	26	35	44	58	71	90	
	−16	−17	−23	−26	−33	−37	−47	
$X_3 =$	0	+1	+1	+1	+2	+1	+1	
				1	1	3	5	9
	−8	−11	−18	−24	−36	−47	−66	−84
	35	44	58	71	90	110	136	163
	−26	−33	−37	−47	−52	−64	−72	−86
$X_4 =$	1	0	+3	+1	+3	+2	+3	+2
			1	1	3	5	9	
	−18	−24	−36	−47	−66	−84	−113	
	71	90	110	136	163	199	235	
	−52	−64	−72	−86	−96	−115	−127	
$X_5 =$	1	+2	+3	+4	+4	+5	+4	
			1	1	3	5	9	13
	−24	−36	−47	−66	−84	−113	−141	−183
	110	136	163	199	235	282	331	391
	−86	−96	−115	−127	−149	−166	−191	−212
$X_6 =$	0	+4	+2	+7	+5	+8	+7	+9
		1	1	3	5	9	13	
	−47	−66	−84	−113	−141	−183	−225	
	199	235	282	331	391	454	532	
	−149	−166	−192	−212	−245	−269	−307	
$X_7 =$	3	+4	+7	+9	+10	+11	+13	
		1	1	3	5	9	13	22
	−66	−84	−113	−141	−183	−225	−284	−344
	282	331	391	454	532	612	709	811
	−212	−245	−269	−307	−338	−382	−419	−472
$X_8 =$	4	+3	+10	+9	+16	+14	+19	+17

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	1	1	3	5	9	13	22	
	− 113	− 141	− 183	− 225	− 284	− 344	− 425	
	454	532	612	709	811	931	1057	
	− 338	− 382	− 419	− 472	− 515	− 576	− 629	
$X_9 =$	4	+ 10	+ 13	+ 17	+ 21	+ 24	+ 25	
	1	1	3	5	9	13	22	30
	− 141	− 183	− 225	− 284	− 344	− 425	− 508	− 617
	612	709	811	931	1057	1206	1360	1540
	− 472	− 515	− 576	− 629	− 699	− 760	− 843	− 913
$X_{10} =$	0	+ 12	+ 13	+ 23	+ 23	+ 34	+ 31	+ 40
	1	3	5	9	13	22	30	
	− 225	− 284	− 344	− 425	− 508	− 617	− 729	
	931	1057	1206	1360	1540	1729	1945	
	− 699	− 760	− 843	− 913	− 1007	− 1091	− 1197	
$X_{11} =$	8	+ 16	+ 24	+ 31	+ 38	+ 43	+ 49	
	1	3	5	9	13	22	30	45
	− 284	− 344	− 425	− 508	− 617	− 729	− 872	− 1020
	1206	1360	1540	1729	1945	2172	2432	2702
	− 913	− 1007	− 1091	− 1197	− 1293	− 1416	− 1525	− 1663
$X_{12} =$	10	+ 12	+ 29	+ 33	+ 48	+ 49	+ 65	+ 64
	3	5	9	13	22	30	45	
	− 425	− 508	− 617	− 729	− 872	− 1020	− 1205	
	1729	1945	2172	2432	2702	3009	3331	
	− 1293	− 1416	− 1525	− 1663	− 1790	− 1945	− 2088	
$X_{13} =$	14	+ 26	+ 39	+ 53	+ 62	+ 74	+ 83	
	3	5	9	13	22	30	45	61
	− 508	− 617	− 729	− 872	− 1020	− 1205	− 1397	− 1632
	2172	2432	2702	3009	3331	3692	4070	4494
	− 1663	− 1790	− 1945	− 2088	− 2265	− 2426	− 2623	− 2807
$X_{14} =$	4	+ 30	+ 37	+ 62	+ 68	+ 91	+ 95	+ 116
	5	9	13	22	30	45	61	
	− 729	− 872	− 1020	− 1205	− 1397	− 1632	− 1877	
	3009	3331	3692	4070	4494	4935	5427	
	− 2265	− 2426	− 2623	− 2807	− 3026	− 3232	− 3479	
$X_{15} =$	20	+ 42	+ 62	+ 80	+ 101	+ 116	+ 132	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	5	9	13	22	30	45	61	85
	−872	−1020	−1205	−1397	−1632	−1877	−2172	−2480
	3692	4070	4494	4935	5427	5942	6510	7104
	−2807	−3026	−3232	−3479	−3708	−3981	−4240	−4541
$X_{16}=$	18	+33	+70	+81	+117	+129	+159	+168
	9	13	22	30	45	61	85	
	−1205	−1397	−1632	−1877	−2172	−2480	−2846	
	4935	5427	5942	6510	7104	7760	8442	
	−3708	−3981	−4240	−4541	−4828	−5164	−5481	
$X_{17}=$	31	+62	+92	+122	+149	+177	+200	
	9	13	22	30	45	61	85	111
	−1397	−1632	−1877	−2172	−2480	−2846	−3228	−3677
	5942	6510	7104	7760	8442	9192	9975	10829
	−4541	−4828	−5164	−5481	−5850	−6204	−6609	−6998
$X_{18}=$	13	+63	+85	+137	+157	+203	+223	+265

CALCULATION OF THE Y'S.

Ind. x even or odd as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	1							
$Y_0=$	1							
				1				
	−1	−1	−1	−1				
$Y_1=$	−1	−1	−1	0	0	0	0	
		1	1	1	2	1	1	
		−1	−1	−2	−2	−2	−2	
						1	1	
					−1	−1	−1	
$Y_3=$	0	0	0	−1	−1	−1	−1	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	1	0	3	1	3	2	3	2
			-1	-2	-3	-5	-5	-5
				1	1	3	2	4
$Y_4 =$	1	0	+2	0	+1	0	0	+1
	1	2	3	4	4	5	4	
	-1	-1	-4	-5	-7	-9	-9	
				1	2	4	6	
						-1	-1	
$Y_5 =$	0	+1	-1	0	-1	-1	0	
		4	2	7	5	8	7	9
		-1	-3	-6	-10	-13	-16	-17
			1	1	5	5	11	10
							-1	-2
$Y_6 =$	0	+3	0	+2	0	0	+1	0
	3	4	7	9	10	11	13	
		-4	-6	-13	-18	-22	-27	
			1	3	7	12	17	
					-1	-1	-4	
$Y_7 =$	3	0	+2	-1	-2	0	-1	
	4	3	10	9	16	14	19	17
		-3	-7	-14	-23	-30	-37	-43
				4	6	17	20	33
						-1	-3	-6
$Y_8 =$	4	0	+3	-1	-1	0	-1	+1
	4	10	13	17	21	24	25	
	-4	-7	-17	-26	-38	-49	-58	
			3	7	17	27	40	
						-4	-6	
$Y_9 =$	0	+3	-1	-2	0	-2	+1	
		12	13	23	23	34	31	40
		-4	-14	-27	-44	-61	-75	-87
			4	7	21	29	52	61
						-3	-7	-14
$Y_{10} =$	0	+8	+3	+3	0	-1	+1	0

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	8	16	24	31	38	43	49	
		-12	-25	-48	-71	-93	-111	
			4	14	31	54	78	
					-4	-7	-17	
$Y_{11} =$	8	$+4$	$+3$	-3	-6	-3	-1	
	10	12	29	33	48	49	65	64
		-8	-24	-48	-79	-109	-136	-161
				12	25	60	84	128
						-4	-14	-27
$Y_{12} =$	10	$+4$	$+5$	-3	-6	-4	-1	$+4$
	14	26	39	53	62	74	83	
	-10	-22	-51	-84	-122	-159	-195	
			8	24	56	95	141	
						-12	-25	
$Y_{13} =$	4	$+4$	-4	-7	-4	-2	$+4$	
	4	30	37	62	68	91	95	116
		-14	-40	-79	-132	-180	-228	-272
			10	22	61	96	161	204
						-8	-24	-48
$Y_{14} =$	4	$+16$	$+7$	$+5$	-3	-1	$+4$	0
	20	42	62	80	101	116	132	
	-4	-34	-71	-133	-197	-258	-316	
			14	40	93	158	233	
					-10	-22	-51	
$Y_{15} =$	16	$+8$	$+5$	-13	-13	-6	-2	
	18	33	70	81	117	129	159	168
		-20	-62	-124	-204	-285	-359	-429
			4	34	75	163	238	350
						-14	-40	-79
$Y_{16} =$	18	$+13$	$+12$	-9	-12	-7	-2	$+10$
	31	62	92	122	149	177	200	
	-18	-51	-121	-202	-301	-397	-486	
			20	62	144	246	367	
					-4	-34	-71	
$Y_{17} =$	13	$+11$	-9	-18	-12	-8	$+10$	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	13	63	85	137	157	203	223	265
		-31	-93	-185	-307	-425	-540	-648
			18	51	139	235	389	511
						-20	-62	-124
$Y_{18} =$	13	$+32$	$+10$	$+3$	-11	-7	$+10$	$+4$

CAMBRIDGE, *December 7th, 1878.*

Remark on the Preceding Paper.

ON discovering the error in Professor Cayley's original statement of the N. G. F. for the seventhic, I caused it to be recalculated out of the grant of the British Association by a method, which will be described in a future communication, considerably shorter than my first method, but somewhat longer than that explained in the text above, perhaps in this instance about half as long again. The table of *Grundformen* obtained by *tamisage* from the corrected N. G. F. table has appeared in the *Comptes Rendus.* The *representative* form in that case is obtained by multiplying numerator and denominator of the N. G. F. fraction by

$$(1 + a^6)(1 + a^{10} + a^{20} + \ldots)(1 + ax)(1 + ax^3)(1 + ax^5),$$

the infinite multiplier being the peculiarity for the seventhic adverted to in the note on the ninthic in this number of the *Journal.* The error in the N. G. F. became apparent from the fact that the sum of the numerical coefficients in the numerator was not equal to zero, a necessary condition, as may easily be proved from and after the case of the quintic. This last, however, only comes into *effectual* operation from the seventhic, because, for the case of the quintic and the sextic, the coefficients consist of pairs of numbers with equal and opposite signs, whereas, for the seventhic and eighthic, the coefficients consist of pairs of equal numbers with the same sign; for the tenthic and eleventhic with opposite signs again and so on, the ratio of the numbers changing by double steps from plus to minus unity.

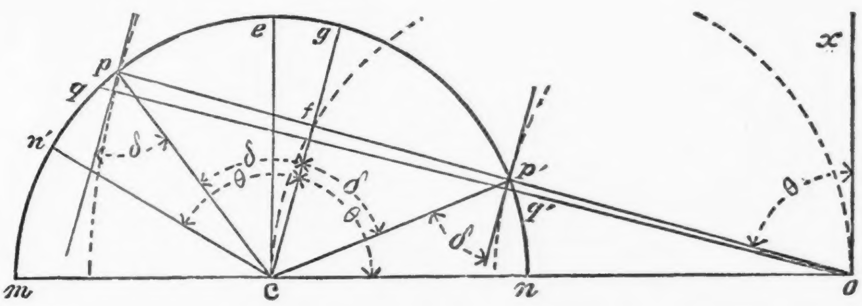
J. J. S.

On the Lateral Deviation of Spherical Projectiles.

By Henry T. Eddy, *Cincinnati, O.*

It may be premised that the phenomenon discussed in this paper is actually observed in pitching base ball; and though it has been often asserted by non-experts that it is impossible to pitch "a curved ball," nevertheless such is not the fact, as appears from careful experiment as well as from the following theoretical investigation. The base ball is selected for experiment in preference to other projectiles, because it is possible to gain more exact information with respect to its initial twist than can be gained as to round shot for example or other spherical projectile.

Since the precise law of resistance which such a projectile experiences in its passage through the air is unknown, we shall be obliged to content ourselves with showing the direction of the deviation without being able at present to compute its numerical amount.



In the figure let c be the center of a spherical projectile whose radius is a, and let *men* be the great circle of the sphere which lies in a horizontal plane. Let us disregard the vertical component of the motion of the projectile; and let c have a horizontal motion of translation, at the instant under consideration, towards e. Also let the projectile have a motion of rotation about a vertical axis through c in a right-handed direction, *i. e.* from m to e. The motions of translation and rotation, whatever be their relative velocities, can be combined, as is well known, into a single motion of rotation about an instantaneous axis parallel to the vertical axis of rotation through c. This

instantaneous axis must intersect the diameter mn, which is perpendicular to the direction of translation ce, at some point, as o. Let the instantaneous axis through o be called the axis of z. Also, let the distance oc be designated by the letter b.

Let r be the shortest (*i. e.* horizontal) distance of any element dS of the surface of the sphere from the axis z. Now pass any vertical plane through the axis of z, cutting the sphere in a circle whose horizontal projection is pp', and similarly pass a second plane qq', making an infinitesimal angle $d\theta = poq$ with pp'; draw cfg perpendicular to pp', and let $\delta = fcp$, $fco = xop = \theta$,

$$\therefore cf = a \cos \delta = b \cos \theta . \qquad (1)$$

In the vertical circle pp' cut out by the first plane, and having fp for its radius, let ϕ be the angle between the radius fp and the radius drawn to any element dS of the surface situated on the circumference of the vertical circle pp'. Then r, θ, ϕ are coordinates of dS, but since the surface is a sphere, we obtain the following relation between these coordinates and known quantities:

$$r = b \sin \theta + a \sin \delta \cos \phi . \qquad (2)$$

Again, since z is the instantaneous axis, the motion of any element dS of the sphere is horizontal and perpendicular to the instantaneous radius r of that element, and, therefore, the relative velocities of different elements are proportional to their respective instantaneous radii r,

$$\therefore v = cr, \qquad (3)$$

in which v is the velocity of any element of the sphere and c is a constant.

Let dS be the quadrilateral element of the spherical surface included between the two planes pp', qq', making an angle $d\theta$ with each other, and two meridian planes intersecting in the line cg and making an angle $d\phi$ with each other. Then is dS ultimately a rectangle, of which the length along the meridian circle is $r \operatorname{cosec} \delta \, d\theta$, and the width along the vertical circle is $a \sin \delta \, d\phi$;

$$\therefore dS = ar \, d\theta \, d\phi . \qquad (4)$$

Disregarding friction, the resultant pressure dP on the element dS, as it moves through the atmosphere, is towards c, and is proportional to v^n (in which the exponent n lies between 1 and 2, but its precise value is unknown), and is also proportional to the cross section $\cos \delta \, dS$ of the stream of air which dS meets in its motion. Then if c' is some constant

$$dP = c'v^n dS = ac'c^n r^{n+1} \cos \delta \, d\theta \, d\phi . \qquad (5)$$

Now resolve dP into two rectangular components; the first, $\sin\delta\, dP$ acting in the plane pp' towards f, the second, $\cos\delta\, dP$ acting perpendicular to the plane pp'.

The first component $\sin\delta\, dP$ has a horizontal component of magnitude $\sin\delta \cos\phi\, dP$, which has a component parallel to co and acting from c toward o, whose magnitude is $\sin\delta \sin\theta \cos\phi\, dP$. The second component $\cos\delta\, dP$ is horizontal, and has a component of magnitude $-\cos\delta\cos\theta\, dP$ acting from c toward o. These are the only components of the normal pressure dP acting along co;

$$\therefore dX = (\sin\delta \sin\theta \cos\phi - \cos\delta\cos\theta)\, dP = \cos\psi\, dP. \quad . \quad . \quad (6)$$

is the horizontal deviating force acting on the element dS, in which ψ is the arc of the great circle joining dS to n', a point so situated in the horizontal plane that the angle $ncn' = 2\theta$. For, let dS be situated at one angle of a spherical triangle of which the remaining two are g and n'; then, since δ and θ are two of its sides and ϕ is the included angle, and ψ is the side opposite ϕ, we have

$$\cos\psi = \sin\delta\sin\theta\cos\phi - \cos\delta\cos\phi, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore dX = bc'c''r^{n+1}\cos\theta\cos\psi\, d\theta\, d\phi. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is readily shown that the deviating force, acting on any elementary ring of the forward half of the sphere included between pp' and qq', is from c toward o, for this deviating force is twice that obtained by integrating (8) with respect to ϕ from $\phi = 0°$ to $\phi = 180°$. And it is possible to show that the value of this integral is a positive quantity, without effecting the integration, by showing that the sum of the positive elements of the integral exceeds the sum of its negative elements. Now it is evident from (2) that, while θ is constant, r decreases as ϕ increases from 0° to 180°, but r never becomes negative in case the axis z lies without the sphere, as, in practice, it does. Again, it appears from the interpretation given to ψ that $\cos\psi$ decreases as ϕ increases from 0° to 180°, but that, so long as $\theta < 90°$, more than half the elements dS along this ring between pp' and qq' are within 90° of n', and hence the largest positive value of $\cos\psi$ numerically exceeds its largest negative value.

Therefore, in integrating (8) with respect to the independent variable ϕ from 0° to 180° (for any value of θ up to 90°), the positive elements of the integral are not only larger but more numerous than its negative elements. And if we afterwards integrate with respect to the independent variable θ from $\theta = \cos^{-1}\frac{a}{b}$ to $\theta = 90°$ (which includes the hemisphere now under

consideration), the total deviating force X will act from c towards o, for each one of its elements will be positive. Furthermore, let us consider the pressures acting upon the remaining hemisphere. These pressures are less than if the projectile stood still, there being a partial vacuum behind it. An experimental comparison of the pressures on the front and back sides of moving bodies shows that the reduction of pressure on the back side below the mean never exceeds about one-half of the increase of pressure on the front side above the mean. Hence, by comparing the pressures on pairs of rings making equal angles with the vertical plane mn, it is seen that, although the deviating force caused by the pressures on the back hemisphere acts from o towards c, it does not numerically exceed about one-half of the deviating force caused by the pressures on the front hemisphere, and acting from c to o. Therefore the total deviating force caused by the normal pressures is from c toward o.

The effect of friction between the air and projectile remains now to be considered. If the air exerted equal pressures at the opposite extremities of each diameter, the friction could diminish the rotary motion, but could cause no deviation. The pressures are not, however, thus distributed. We may state the case roughly thus: in the quarter of the sphere projected on the paper in mec the average pressures are greater than in either other quarter, the average pressures in ecn are next largest; and, if e' is the other extremity of a diameter through e, the average pressures in nce' are next largest, and those in $e'cm$ are the smallest.

The difference of the pressures in the opposite quarters mce, nce' causes a difference of frictions; the same difference exists between the opposite quarters ecn, $e'cm$; and these two effects do not differ greatly in magnitude. Hence the total effect might be replaced by a friction at a point of the surface not far from e. It is evident that the effect of applying friction at such a point would be to cause the projectile to roll away from o, so that the component of the deviating force furnished by the friction is from o to c. But the amount of this force is inconsiderable compared with that caused by the differences of the normal pressures, being dependent, however, upon the roughness of the surface of the projectile. Rankine* states that Smeaton's experiments show that the coefficient of friction for the best sails of wind-mills is probably about 0.016.

* Manual of the Steam Engine, etc. 7th ed. London, 1874, chap. VIII.

Note on Determinants and Duadic Disynthemes.

By J. J. Sylvester.

A general algebraical determinant in its developed form (viewed in relation to any one arbitrarily selected term) may be likened to a mixture of liquids seemingly homogeneous, but which being of differing boiling points, admit of being separated by the process of fractional distillation. Thus *ex. gr.* suppose a general determinant of the 6th order. The 720 terms which make it up will fall, in relation to the leading diagonal product, into as many classes (most of which comprise several similarly constituted families) as there are unlimited partitions of 6. These, 11 in number, are

6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 3, 1, 1, 1; 2, 2, 2; 2, 2, 1, 1; 2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1.

Let the determinant be represented, in the umbral notation, by

$$\begin{matrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{matrix}\,.^{*}$$

Let us, by way of illustration, consider the class corresponding to 6; this will consist of the 1.2.3.4.5 (120) terms obtained by forming the 120 distinct circular arrangements that belong to $a\,b\,c\,d\,e\,f$. Thus:

$$\begin{matrix} & a & & c & \\ b & & & & e \\ & f & & d & \end{matrix}$$

will signify $ac \times ce \times ed \times df \times fb \times ba$, which will be one of the 120 in question. So, again, 3, 3 will denote, in the first place, the 10 sets of double triads of the general form $abc : def$, and, as each triad will give two cyclical orders, there will in all be 10×2^2, *i. e.* 40, terms of the form $ab\,.\,bc\,.\,ca\,.\,de\,.\,ef\,.\,fd$. So, again, there will be 15.1^3, *i. e.* 15, corresponding to 2, 2, 2. So 3, 2, 1 will give 10 groupings of the form $abc : de : f$, and each of these will give rise to two

* The cyclical method of the text shows what was not previously apparent, that the umbral notation $\begin{matrix} ab\ldots l \\ ab\ldots l\end{matrix}$ possesses an essential advantage over $\begin{matrix} ab\ldots l \\ \alpha\beta\ldots\lambda\end{matrix}$ even for unsymmetrical determinants. This mode of notation of course implies some ground of preference for one diagonal group over all others and thus virtually regards a general determinant as related to a lineo-linear as a symmetrical one is to a quadratic form. For instance the general determinant of the second order is to be regarded as appurtenant to the lineo-linear form $aaxx' + abxy' + bayx' + bbyy'$.

terms, viz: $ab\,.\,bc\,.\,ca\,.\,de\,.\,ed\,.\,ff$, $ac\,.\,cb\,.\,ba\,.\,de\,.\,ed\,.\,ff$, the number of cycles corresponding to two elements de being 1, and to one element f also 1.

This simple theory affords us a direct means of calculating the number of distinct terms in a symmetrical determinant, *i. e.* one in which $i\,.\,j$ and $j\,.\,i$ are identical. It enables us to see at once that the coefficient of every term is unity or a power of 2; the rule being that plus or minus terms* of the class corresponding to $m_1, m_2, m_3, \ldots$ will take the coefficient 2^ν, ν being the number of the quantities m which are neither 1 nor 2, for, in every other case, the total number of cycles in each partial group will arrange themselves in pairs which give the same result, thus *ex. gr.*

$$\begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array}$$

will give the equal products $ab\,.\,bc\,.\,cd\,.\,da$ and $ad\,.\,dc\,.\,cb\,.\,ba$.

As an example of the direct method of computation, take a symmetrical determinant of the 5th order. Write

$$5 \quad 4.1 \quad 3.2 \quad 3.1.1 \quad 2.2.1 \quad 2.1.1.1 \quad 1.1.1.1.1.$$

To these 7 classes there will belong respectively

1.12	with the	coefficient	2
5.3	"	"	2
10.1	"	"	2
10.1	"	"	2
15	"	"	1
10	"	"	1
1	"	"	1.

Thus the number of distinct terms will be

$$12 + 15 + 10 + 10 + 15 + 10 + 1 = 73,$$

and the sum of the coefficients

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120,$$

both of which are right.

Again, if we have a skew determinant of an even order, it will easily be seen that any partition embracing one or more odd numbers will give rise to pairs of terms that mutually cancel, but when all the parts into which the exponent of the order is divided are even, the coefficient will be given by the same rule as for symmetrical determinants, *i. e.* its arithmetical value will be 2^ν, where ν is the number of parts exceeding 2. Thus *ex. gr.* for a skew determinant of the order 6 we have

$$6 \quad 4.2 \quad 2.2.2.$$

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*The complete value of the coefficient is $(-)^\mu 2^\nu$, ν being the number of elements in the partition other than 1 or 2, and μ the number of even elements.

The number of terms corresponding to these partitions being 60 with coefficient 2, 15×3 also with coefficient 2, and 15 with coefficient 1, making 120 distinct terms in all, the sum of the coefficients will be

$$120 + 90 + 15 = (1.3.5)^2,$$

which is right, because the result is the square of the sum of 15 synthemes of the form $1.2 \times 3.4 \times 5.6$. It may be observed that 120 is $\frac{15.16}{2}$, as it ought to be, because, until we reach the order 8, the same *double duadic syntheme* can only be made up in one way of two simple ones, but this ceases to be the case from and after 8. Thus *ex. gr.* the pair of synthemes

$$1.2 \quad 3.4 \quad 5.6 \quad 7.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.7 \quad 6.8$$

combined will produce the same double syntheme as the pair

$$1.2 \quad 3.4 \quad 5.7 \quad 6.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.6 \quad 7.8,$$

and accordingly for 8 we have the partitions

$$8 \quad 6.2 \quad 4.4 \quad 4.2.2 \quad 2.2.2.2,$$

giving rise to

2520	with	coefficient	2
28.60	"	"	2
35.3^2	"	"	4
210.3	"	"	2
105	"	"	1,

making in all $2520 + 1680 + 315 + 630 + 105$, *i. e.* 5250, distinct terms, whereas,

$$\frac{(1.3.5.7)^2 + (1.3.5.7)}{2} = 5565,$$

the difference, 315, being due to the fact that there are that number of double synthemes which admit of a twofold resolution into two single synthemes.

I will not stop to prove, but any person conversant with the subject will see at once that this method gives an intuitive and direct proof of the theorem that a pure skew determinant for an even order is a perfect square.* Having only a limited space at my command, I will pass on at once to forming the equation in differences for the case of a symmetrical, a skew, and one or two other special forms of determinants.

1°. For a symmetrical determinant, taking as a diagram, to fix the ideas, the matrix of the 6th order

$$\begin{matrix} a & b & c & d & e & f \\ b & g & h & k & l & m \\ c & h & n & p & q & r \\ d & k & p & s & t & u \\ e & l & q & t & v & w \\ f & m & r & u & w & \omega \end{matrix},$$

* That a skew determinant of an odd order vanishes is apparent from the fact that an odd number cannot be made up of a set of even ones. I use the term skew determinant in its strict sense as referring to a matrix for which $ij = -ji$ and $ii = 0$.

calling u_m the number of distinct terms in a symmetrical matrix of the mth order, and, resolving the entire determinant into a sum of determinants of the order $(m-1)$ multiplied by the letters in the top line, we shall obviously get u_{m-1} together with $(m-1)$ quantities, positive or negative (and we know, by what precedes, that there can be no canceling, so that the sign, for the object in view, may be entirely neglected) of the form

$$\begin{array}{cccccc} & b & h & k & l & m \\ & c & n & p & q & r \\ b \times & d & p & s & t & u\,. \\ & e & q & t & v & w \\ & f & r & u & w & \omega \end{array}$$

Among these $(m-1)$ quantities all the terms containing bc, bd, be, bf will occur twice over, but those containing b^2 do not recur. Hence, to find the number of distinct terms we may reckon each of such distinct terms as contain bc, bd, be, bf worth only $\frac{1}{2}$, the others counting as 1. But if, instead of the column (which I write as a line) $bcdef$, we had the column $bhklm$, the rule for calculating the number of distinct terms might be calculated by this very same rule, except that the terms multiplied by hc, kd, le, mf ought to count as *units* instead of *halves*. Hence obviously

$$u_m + (m-1)(m-2)\, u_{m-3} \times \frac{1}{2} = u_{m-1} + (m-1)\, u_{m-1} = m u_{m-1},$$

or

$$u_m = m u_{m-1} - \frac{(m-1)(m-2)}{2}\, u_{m-3},$$

which is Mr. Cayley's equation, but obtained by a much more expeditious process (see Salmon's Higher Algebra, 3d edition, pp. 40–42); writing $u_m = (1\,.\,2\,.\,.\,.\,m)\, v_m$ we obtain the equation in differences, linear in regard to the independent variable,

$$m v_m - m v_{m-1} + \frac{v_{m-3}}{2} = 0,$$

and this, treated by the general method applicable to all such, gives rise to a linear differential equation in which, on account of the particular initial values of u_0, u_1, u_2, the third term is wanting, and finally v_m is found to be the coefficient of t^m in

$$\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}},$$

If we apply a similar method to the case of a symmetrical determinant in which the diagonal of symmetry is filled out with zeros (an invertebrate symmetrical or symmetrical bialar determinant, as we may call it) we shall easily obtain the equation in differences

$$u_m = (m-1)\,[u_{m-1} + u_{m-2}] - \frac{(m-1)(m-2)}{2}\,u_{m-3},$$

and, making $u_m = 1.2\ldots mv_m$,

$$mv_m - (m-1)\,v_{m-1} - v_{m-2} + \frac{u_{m-3}}{2} = 0,$$

from which, calling $y = v_0 + v_1 t + v_2 t^2 + \ldots$ and having regard to the initial values v_0, v_1, v_2, we obtain

$$2\frac{dy}{y} = \frac{2t - t^2}{1-t}\,dt,$$

and

$$y = \frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}.$$

By way of distinction, using u' and v' for this case, and u, v for the preceding one, the slightest consideration shows that

$$u_m = u'_m + mu'_{m-1} + \frac{m(m-1)}{2}\,u'_{m-2} + \frac{m(m-1)(m-2)}{2.3}\,u'_{m-3} + \ldots,$$

or

$$v_m = v'_m + v'_{m-1} + \frac{v'_{m-2}}{1.2} + \frac{v'_{m-3}}{1.2.3} + \ldots.$$

Hence the generating function for v_m ought to be that for u_m multiplied by e^t, as we see is the case.

So, in like manner, the generating function for v_m, *i. e.* $\frac{u_m}{1.2\ldots m}$, in the case of a general determinant being $\frac{1}{1-t}$, that of v_m for an invertebrate or zero-axial but otherwise general determinant we see must be $\frac{e^{-t}}{1-t}$*, *i. e.*

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* It may easily be proved that the difference between the numbers of positive and negative combinations in the development of an invertebrate determinant of the mth order is $(-)^{m-1}(m-1)$ in favor of the former. From this it is easy to prove that the generating function for $\frac{\text{number of positive terms in such determinant}}{1.2.3\ldots m}$ is

$$\frac{1}{2}\left\{\frac{e^{-t}}{1-t} - (1+t)\,e^{-t}\right\}, \text{ or } \frac{t^2e^{-t}}{2(1-t)}.$$

Whence it follows that the number of positive terms in a general invertebrate determinant of the mth order is $m\frac{m-1}{2}$ times the total number of the terms in one of the $(m-2)$th order. The equation of differences for U_m, the total number, is of course

$$U_m = (m-1)(U_{m-1} + U_{m-2}),$$

and the successive values of

U_m for	1,	2,	3,	4,	5,	6,	7,	8,	....
are	0,	1,	2,	9,	44,	265,	1854,	14833,	....

$$v_m = 1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \ldots \pm \frac{1}{1.2\ldots.m},$$

the well known value $\left(\text{ultimately equal to } \frac{1}{e}\right)$, as it ought obviously to be, of the chance of two cards of the same name not coming together when one pack of m distinct cards is laid card for card under another precisely similar pack.

Returning to the case of the invertebrate symmetrical determinant, it will readily be seen, by virtue of the prolegomena, that the number of terms (the u_m) for such a determinant of the mth order is the same thing as the total number of duadic disynthemes that can be formed with m things, meaning by a duadic disyntheme any combination of duads with or without repetition, in which each element occurs twice and no oftener. Thus, when $m = 6$, $1.2\ 2.3\ 1.3\ 4.5\ 4.6\ 5.6$ and $1.2\ 2.3\ 3.4\ 5.6\ 6.1$ and $1.2\ 2.3\ 3.4\ 1.4\ 5.6\ 5.6$ are all three of them disynthemes. But the two latter ones are each resoluble into single synthemes, whereas the first one is not. It is clear that, when a disyntheme is formed by means of cycles all of an even order, it will be resoluble into a pair of single synthemes, and in no other case. The problem, then, of finding the number of distinct double synthemes with m elements is one and the same as that of finding the number of distinct terms in a *proper* (*i. e.* invertebrate) skew determinant, which I proceed to consider.

Following a method (not identical with but) analogous to that adopted for the symmetrical cases, we shall find, by a process which the terms below written will sufficiently suggest

$$u_m + \frac{(m-1)(m-2)(m-3)}{2} u_{m-4} = (m-1)\, u_{m-2} + (m-1)(m-2)\, u_{m-2},$$

or

$$u_m = (m-1)^2\, u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}.$$

Of course, when m is odd $u_m = 0$. From this it is readily seen that $\frac{u_{2m}}{1.3.5\ldots 2_{m-1}}$, say ω_m, is an integer; for we shall have

$$\omega_m = (2m-1)\, \omega_{m-1} - (m-1)\, \omega_{m-2},$$

also,

$$\omega_1 = 1, \quad \omega_2 = 2,$$

so that

$$\begin{aligned} \omega_3 &= 5.2 - 2.1 = 8, \\ \omega_4 &= 7.8 - 3.2 = 50, \\ \omega_5 &= 9.50 - 4.8 = 418, \\ \omega_6 &= 11.418 - 5.50 = 4348, \end{aligned}$$

and the conventional $\omega_0 = 3\omega_1 - \omega_2 = 1$.

By the above formula u_m can be calculated with prodigious rapidity. If, however, we wish to obtain a generating function for u_m, the differential equation obtained from the above equation in differences does not lead to a simple explicit integral, but if we make $u_{2m} = (1.2.3\ldots 2m)\, v_m$, as in the preceding cases, or, which is the same thing, $\omega_m = 2_m\,(1.2\ldots m)\, v_m$, we get

$$4mv_m - 4(m-1)\, v_{m-1} - 2v_{m-1} + v_{m-2} = 0,$$

and, writing as before $y = v_0 + v_1 t + v_2 t^2 + \ldots,$

$$4\frac{dy}{dt} - 4t\frac{dy}{dt} - 2y + ty$$

will be found to be equal to zero. [This vanishing of the 3d term in the differential equation being a feature common to all the cases we have considered, and due to the initial values of the v series in each case.] We have thus

$$\frac{4y'}{y} = \frac{1}{1-t} + 1, \quad y = \frac{e^{\frac{t}{4}}}{(1-t)^{\frac{1}{4}}}.$$

By way of verification, we may observe that

$$v_0 = 1, \quad v_1 = \frac{1}{2}, \quad v_2 = \frac{1}{4}, \quad v_3 = \frac{1}{6}, \ldots,$$

$$y = \left(1 + \frac{t}{4} + \frac{t^2}{32} + \frac{t^3}{384} + \ldots\right)\left(1 + \frac{t}{4} + \frac{5t^2}{32} + \frac{45t^3}{384} + \ldots\right),$$

and

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \frac{1}{32} + \frac{1}{16} + \frac{5}{32} = \frac{1}{4}, \quad \frac{45}{384} + \frac{5}{128} + \frac{1}{128} + \frac{1}{384} = \frac{1}{6}.\text{*}$$

We may now proceed to calculate the number of distinct terms in an improper or vertebrated skew-determinant, which is interesting on account of its connection with the theory of orthogonal transformations. Using v_{2m}, instead of v_m, the generating function for the case last considered becomes $\dfrac{e^{\frac{t^2}{4}}}{\sqrt[4]{1-t^2}}$. Let $(1.2.3\ldots m)\, V_m = U_m$ in general be used to denote the number of distinct terms in a vertebrate skew-determinant of the mth order. Then obviously

$$U_{2m} = u_{2m} + m.\frac{m-1}{2}\, u_{2m-2} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}\, u_{2m-4} + \ldots,$$

or

$$V_{2m} = v_{2m} + \frac{v_{2m-2}}{1.2} + \frac{v_{2m-4}}{1.2.3.4} + \ldots.$$

Hence the generating function for V_{2m}

$$= \frac{e^{\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\left\{1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \ldots\right\} = \frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} + e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\},$$

* The values of $v_1, v_2, v_3\ldots$ are $\frac{1}{2}, \frac{2}{2.4}, \frac{8}{2.4.6}, \frac{50}{2.4.6.8}, \ldots$; *i. e.* $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{25}{192}, \ldots$

and in like manner, since

$$U_{2m-1} = mu_{2m-2} + \frac{m(m-1)(m-2)}{1.2.3} u_{2m-4} + \ldots,$$

the generating function for V_{2m-1} will be

$$\frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} - e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\}.$$

Hence the number of distinct cross-products in the development of an orthogonal transformation-matrix of the mth order is

$$(1.2.3\ldots m) \times \text{coefficient of } t^m \text{ in } \frac{e^{t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}.$$

POSTSCRIPT.—Let us consider the case of $2m$ elements; call the number of ways in which any disyntheme composed with them may be resolved into a pair of single synthemes one in each hand* its weight; furthermore, call the aggregate of those which appertain to an odd number of cycles the first class, and the other the second class. The entire sum of the weights we know is $1^2.2^2.3^2\ldots\overline{2m-1}^2$, but, furthermore, I find that the excess of the total weight of the first class over that of the second is $1^2.2^2.3^2\ldots\overline{2m-3}^2.\overline{2m-1}$; or, in other words, the weights of the two classes are in the ratio of m to $m-1$.

The expressions for the sum and for the difference may, of course, by the *prolegomena* be translated into two theorems on the partition of numbers, neither of which, as far as I can see, is obvious upon the face of it.†

* The two hands are introduced in order to double, by the effect of permutation, what the weight otherwise would be, except when the two component synthemes are identical, in which case the weight remains unity.

† REMARK.—The equation in differences for the number of double duadic synthemes may be obtained without recourse to determinants, as follows: Single out any element, 1; it may be paired in each of the component synthemes with any one of the remaining elements 2, 3, 4, . . . , and there are two cases to be distinguished, viz: 1 may be paired either with the same element (2) or with two different elements (2, 3), in the two synthemes. The former may be done in $(m-1)$ ways, and, after having made our choice, we have still the choice of all the double synthemes that can be formed from $3, 4, \ldots m;\ 3, 4, \ldots m$. The choice of two *different* elements may be made in $\frac{(m-1)(m-2)}{2}$ ways, and having chosen, we have still the choice of all the double synthemes that can be formed from $3, 4, \ldots m;\ 2, 4, \ldots m$. Now it is plain that the number of these can be obtained from the number of double synthemes that can be formed from $3, 4, \ldots m;\ 3, 4, \ldots m$, by counting twice all except those in which 3 is paired twice with the same element; and is equal, therefore, from what precedes, to

$$2u_{m-2} - (m-3)\,u_{m-4}.$$

We have, therefore,

$$\begin{aligned} u_m &= (m-1)\,u_{m-2} + \frac{(m-1)(m-2)}{2}\,[2u_{m-2} - (m-3)\,u_{m-4}] \\ &= (m-1)^2\,u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2}\,u_{m-4}. \end{aligned}$$

F. FRANKLIN.

Desiderata and Suggestions.

By Professor Cayley, *Cambridge, England.*

No. 3.—THE NEWTON-FOURIER IMAGINARY PROBLEM.

The Newtonian method as completed by Fourier, or say the Newton-Fourier method, for the solution of a numerical equation by successive approximations, relates to an equation $f(x) = 0$, with real coefficients, and to the determination of a certain real root thereof a by means of an assumed approximate real value ξ satisfying prescribed conditions: we then, from ξ, derive a nearer approximate value ξ_1 by the formula $\xi_1 = \xi - \dfrac{f(\xi)}{f'(\xi)}$; and thence, in like manner, $\xi_1, \xi_2, \xi_3, \ldots$ approximating more and more nearly to the required root a.

In connexion herewith, throwing aside the restrictions as to reality, we have what I call the Newton-Fourier Imaginary Problem, as follows.

Take $f(u)$, a given rational and integral function of u, with real or imaginary coefficients; ξ, a given real or imaginary value, and from this derive ξ_1 by the formula $\xi_1 = \xi - \dfrac{f(\xi)}{f'(\xi)}$, and thence $\xi_1, \xi_2, \xi_3, \ldots$ each from the preceding one by the like formula.

A given imaginary quantity $x + iy$ may be represented by a point the coordinates of which are (x, y): the roots of the equation are thus represented by given points $A, B, C \ldots$, and the values $\xi, \xi_1, \xi_2 \ldots$ by points $P, P_1, P_2, \ldots$ the first of which is assumed at pleasure, and the others each from the preceding one by the like given geometrical construction. The problem is to determine the regions of the plane, such that P being taken at pleasure anywhere within one region we arrive ultimately at the point A; anywhere within another region at the point B; and so for the several points representing the roots of the equation.

The solution is easy and elegant in the case of a quadric equation, but the next succeeding case of the cubic equation appears to present considerable difficulty.

Cambridge, *March 3d, 1879.*

On the Complete System of the "Grundformen" of the Binary Quantic of the Ninth Order.

By J. J. Sylvester.

Enumeration of the Irreducible Invariants and Covariants of the Binary Quantic of the Ninth Order.

	Order in the Variables.																				
Degree in the Coefficients.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

The foregoing table has been calculated, out of the funds voted by the British Association, under my superintendence, by Mr. Franklin, Fellow of Johns Hopkins University. A statement of the method employed will be given in a future number of the *Journal.*

The total number of irreducible forms will be seen from the table to be 415. The highest degree in the coefficients is 18, and the highest order in

the variables 22. The *representative* generating function in this case (as in all others which have been hitherto treated, with the sole exception of the seventhic) has a *finite* numerator.

The total number of groundforms for the orders 0, 2, 4, 6 respectively (counting, as one ought to do, the absolute constant as one of them) is 1, 3, 6, 27, which becomes a regular series on increasing 6, which corresponds to a square index 4, in the proportion of 2 : 3. In like manner, for the orders 1, 3, 5, 7, 9, the series is 2, 5, 24, 125, 416, which, on increasing the last term corresponding to the square index 9 in the ratio 2 : 3, forms an almost regular progression 2, 5, 24, 125, 624, highly suggestive of the geometrical series 1, 5, 25, 125, 625. It seems then to be a not altogether improbable conjecture, that the number of groundforms for 10, which I hope very soon to get completely worked out, will be in the neighbourhood of a ratio of equality to 243,* and for 11, which there is not much prospect of calculating for some time to come, a number not very far out from a ratio of equality to 3125. In the next, or next but one, number of the *Journal* I hope to set out a synoptical table of the groundforms for all orders up to 10 inclusive, with their reduced and representative generating functions, as also for combinations of the orders: 2, 3; 2, 4; 3, 3; 3, 4; 4, 4; all the materials for which, with the exception of what pertains to the covariants *proper* of the tenthic, are already in existence.

Extract of a Letter from Sig. A. de Gasparis to Mr. Sylvester.

... J'ai trouvé certaines séries dans lesquelles les éléments tels que le rayon vecteur, les anomalies excentriques et vraies, etc., sont exprimés en fonction de l'anomalie moyenne *donnée en parties du rayon* sans sinus ou cosinus. Comme essai, je vous comunique les suivantes dans lesquelles $e =$ excentricité, v et M anomalie vraie et moyenne. En outre $a =$ demigrand axe, i, ϕ inclinaison et noend, π perihelie, $\psi = \pi - \phi$. J'ai trouvé

$$v = \sqrt{\frac{1+e}{1-e}} \left\{ \frac{M}{1-e} - \frac{M^3}{6} \frac{2e}{(1-e)^4} + \frac{M^5}{120} \frac{2e+2ae^2}{(1-e)^7} - \frac{M^7}{5040} \frac{18e+22e^2-900e^3}{(1-e)^{10}} + \dots \right\};$$

et posant

$$H = (1-e)\sin\psi + \frac{M}{1}\sqrt{\frac{1+e}{1-e}}\cos\psi - \frac{M^2}{2}\frac{\sin\psi}{(1-e)^2} - \frac{M^3}{6}\sqrt{\frac{1+e}{1-e}}\frac{\cos\psi}{(1-e)^3}$$

* The number of groundforms for the Octavic (I quote from memory) is 70, not more inferior to 81 than might have been anticipated, when the composite form of the number 8 is taken into account. It seems likely that for 10, 243 is at all events a superior limit.

$$+\frac{M^4}{24}(1+3e)\frac{\sin\psi}{(1-e)^5}+\frac{M^5}{120}\sqrt{\frac{1+e}{1-e}}(1+9e)\frac{\cos\psi}{(1-e)^6}-\ldots,$$

$$K=(1-e)\cos\psi-\frac{M}{1}\sqrt{\frac{1+e}{1-e}}\sin\psi-\frac{M^2}{2}\frac{\cos\psi}{(1-e)^2}+\frac{M^3}{6}\sqrt{\frac{1+e}{1-e}}\frac{\sin\psi}{(1-e)^3}$$

$$+\frac{M^4}{24}(1+3e)\frac{\cos\psi}{(1-e)^5}-\frac{M^5}{120}\sqrt{\frac{1+e}{1-e}}(1+9e)\frac{\sin\psi}{(1-e)^6}-\ldots,$$

l'on a pour la valeur des coordonnées héliocentriques z, x et y

$$\frac{z}{a}=\sin i.H;\quad \frac{x}{a}=\cos\phi.K-\sin\phi\cos i.H;\quad \frac{y}{a}=\sin\phi.K+\cos\phi\cos i.H.$$

L'on peut aussi développer, comme j'ai fait jusqu' aux cubes de M, la valeur inverse du cube de la distance mutuelle des deux masses, telle que se présente dans la théorie des perturbations. Dans ce cas figurent les deux variables M_1 et M_2. Par la relation linéaire qui existe entre le temps et l'anomalie moyenne il peut etre utile de considerer ces developpements dans le calcul des perturbations.

J'ai publié aussi dans les actes des academies des Lincei et de Naples, le coefficient du terme qui multiplie la 4$^{\text{ème}}$ puissance du temps dans la série qui donne la correction de la coordonnée x *elliptique* pour avoir la valeur de la x troublée dans *le temps T après le temps t* pour lequel on connait les x, y, z et leurs derivées x', y', z', étant m_1 la masse troublée, et m_2 la masse troublante. Ce coefficient, sauf un facteur connu de l'ordre m_2, est

$$+\frac{x_1}{r_1^3\rho_{12}^3}-\frac{x_2}{r_2^3\rho_{12}^3}-\frac{(m_1+m_2)(x_2-x_1)}{\rho_{12}^6}+\frac{(1+m_2)x_2}{r_2^6}+\frac{m_1x_1}{r_1^3r_2^3}-\frac{6(x'_2-x'_1)\rho'_{12}}{\rho_{12}^4}$$

$$+\frac{15(x_2-x_1)\rho'^2_{12}}{\rho_{12}^5}+\frac{6x'_2r'_2}{r_2^4}-\frac{3(x_2-x_1)^2}{\rho_{12}^5}\left\{\frac{x_1}{r_1^3}-\frac{x_2}{r_2^3}-\frac{(m_1+m_2)(x_2-x_1)}{\rho_{12}^3}\right\}$$

$$-\frac{3(x_2-x_1)(y_2-y_1)}{\rho_{12}^5}\left\{\frac{y_1}{r_1^3}-\frac{y_2}{r_2^3}-\frac{(m_1+m_2)(y_2-y_1)}{\rho_{12}^3}\right\}$$

$$-\frac{3(x_2-x_1)(z_2-z_1)}{\rho_{12}^5}\left\{\frac{z_1}{r_1^3}-\frac{z_2}{r_2^3}-\frac{(m_1+m_2)(z_2-z_1)}{\rho_{12}^3}\right\}$$

$$-\frac{3x_2^2}{r_2^5}\left\{\frac{m_1(x_2-x_1)}{\rho_{12}^3}+\frac{(1+m_2)x_2}{r_2^3}+\frac{m_1x_1}{r_1^3}\right\}-\frac{3x_2y_2}{r_2^5}\left\{\frac{m_1(y_2-y_1)}{\rho_{12}^3}+\frac{(1+m_2)y_2}{r_2^3}+\frac{m_1y_1}{r_1^3}\right\}$$

$$-\frac{3x_2z_2}{r_2^5}\left\{\frac{m_1(z_2-z_1)}{\rho_{12}^3}+\frac{(1+m_2)z_2}{r_2^3}+\frac{m_1z_1}{r_1^3}\right\}-\frac{3(x_2-x_1)}{\rho_{12}^5}\left\{(x'_2-x'_1)^2+(y'_2-y'_1)^2+(z'_2-z'_1)^2\right\}$$

$$+\frac{m_1(x_2-x_1)}{r_2^3\rho_{12}^3}-\frac{12x_2r'^2_2}{r_2^5}+\frac{3x_2}{r_2^5}(x'^2_2+y'^2_2+z'^2_2-r'^2_2).$$

NAPLES, *15 Mars, 1879.*

An Essay on the Calculus of Enlargement.

BY EMORY MCCLINTOCK, F. I. A., *Actuary of the Northwestern Mutual Life Insurance Company, Milwaukee, Wisconsin.*

CONTENTS.

A. OUTLINE.

1. The Calculus of Enlargement is, from one point of view, an extension of the Calculus of Finite Differences; from another, an extension of the Calculus of Operations. It comprises, as its most important branch, the Differential Calculus, included in which is the Calculus of Variations. The scope of this new science is, therefore, comprehensive. Its method, on the other hand, is simple. My present object is not to exhibit it in a methodical treatise, but merely to give a preliminary sketch of it, and so to publish its discovery.

2. The Calculus of Enlargement is, from one point of view, an extension of the Calculus of Finite Differences. It has for its basis the well-known operation $\text{E} = 1 + \Delta$, or rather, as I prefer to state it, the operation E^h, where

$$\text{E}^h x = x + h, \tag{1}$$

$$\text{E}^h \phi(x) = \phi(x + h). \tag{2}$$

I call this operation Enlargement.*

*The term is elliptical, since by the Enlargement of a function is meant that change which results from the enlargement of the variable. It would probably be hard, however, to find a more appropriate name. The word Enlargement has this further advantage, that its initial letter has been long in use as the symbol of this operation. It will, of course, sometimes be necessary to call that a negative enlargement which is in reality a diminution, just as the word increment sometimes denotes that which is arithmetically a decrement.

3. From another point of view, the Calculus of Enlargement is a modification and extension of the Calculus of Operations, or doctrine of the Separation of Symbols. The symbolic method, as usually explained, concerns itself with the symbol of differentiation, $\frac{d}{dx}$ or D, and with the various functions of that symbol, considered apart from the subject of operation; and among these functions of D is $E = \varepsilon^D$. The Calculus of Enlargement, on the other hand, regards E as the fundamental symbol, and takes cognizance of other symbols only in case they are, and because they are, functions* of E. Among such, of course, is $D = \log E$. If we conceive the symbolic method to be modified and defined in this manner, and to be ranked as a science by itself, instead of a mere auxiliary principle; and if we further conceive this science to be so extended as to include not only, as at present, the separate treatment of symbols of operation, but also a complete discussion of the operations denoted by such symbols, their definitions, uses and consequences, we shall have in mind the Calculus of Enlargement.†

4. The theory of differentiation, comprising the Differential and Integral Calculus and their applications, and including the Calculus of Variations, of which the fundamental operation is differentiation with respect to an imagined variable, forms the most important branch of the Calculus of Enlargement. The algebra of the functions of E is subject to all the laws of ordinary algebra; and *the theory of differentiation is that part of the calculus which corresponds to the theory of logarithms in algebra.*

5. In this manner is effected the orderly unification of those branches of science which I have mentioned. Writers on finite differences have said repeatedly that a differential is but a certain kind of difference, so that the differential calculus may be regarded as a part of the former science; but the connection thus indicated is so trivial, and its consequences are so insignificant, that the claim excites no attention. Nevertheless, it will be agreed that the boundary line between these two branches is but indistinct, and that their formal union, supposing it to be accomplished in a natural and simple manner, is a result to be desired. The obvious connecting link is the equation

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*By "function" of x, throughout this essay, I mean a quantity which can be expressed by a series of terms, each of the form Ax^a, where A and a are independent of x, and are not necessarily integral or positive.

†"This branch of science [the Calculus of Operations] is yet in its infancy, but already it has been the instrument of greatly extending the domains of science, and we may reasonably look to it for the next great step in the direction of mathematical progress."—Davies & Peck, *Mathematical Dictionary*, p. 401.

$\text{E} = \varepsilon^{\text{D}}$, or its converse, $\text{D} = \log \text{E}$. The union must be effected, if at all, in one of two ways. On the one hand, we may begin by defining $\frac{d}{dx}$ or D, and then proceed to $\text{E} = \varepsilon^{\text{D}}$ and $\Delta = \varepsilon^{\text{D}} - 1$. This is the unnatural order hitherto tacitly followed, not only by those writers who have appended a chapter or two on finite differences to their treatises on the differential calculus, but also in works devoted to finite differences, all of which, in late years, assume a prior knowledge of differentiation. A student is first taught differentiation; later, he learns the doctrine of the separation of symbols, and finally, if sufficiently zealous, he takes up finite differences. In the latest book on this subject, that of Boole, the reader is referred, for the readiest proof that D, E, and Δ are mutually subject to algebraic discussion, to a passage in that author's work on differential equations. We may, on the other hand, adopt the more natural order, defining E first, and giving afterwards, as one of its functions,

$$\text{D} = \log \text{E}. \tag{3}$$

This well-known equation has not hitherto, I believe, been proposed as the definition of the symbol, and therefore of the operation, of differentiation. To say that Differentiation is the logarithm of Enlargement would seem, and possibly be, a quasi-metaphysical absurdity; but we can and should say that Differentiation is that operation whose symbol is the logarithm of the symbol of Enlargement. Of the two operations, the simpler should be defined the earlier. Now

$$\text{E}\phi(x) = \phi(x+1) \tag{4}$$

is a simpler statement than

$$\text{D}\phi(x) = \frac{\varphi(x+h) - \varphi(x)}{h}_{[h=0]}. \tag{5}$$

These operations, E and D, are functions of each other, and whichever is defined last must be expressed in terms of the other. That D shall be defined in terms of E is the most important feature of the Calculus of Enlargement.

6. The theory of differentiation, I have said, is that part of the calculus which corresponds to the theory of logarithms in algebra. This proposition leads directly to very important consequences. Since D is a function of E, all theorems which may be discovered concerning $\phi(\text{E})$ will be true of D, and also, more generally, true of $\psi(\text{D})$, supposing $\phi(x) = \psi(\log x)$. I shall show that in this manner the known theorems of the differential calculus can be proved, and novel truths discovered, by a method almost startling from its simplicity. Again, from every known or ascertainable proposition in the

theory of logarithms we shall derive at once a corresponding proposition in the theory of differentiation; while, conversely, additions will be made to the theory of logarithms analogous to known truths in that of differentiation. Finally, from every known or ascertainable equation representing $\log x$ in terms of x, or of any simple function of x, we shall derive a corresponding explanation, or practical definition, of the operation of differentiation; including not only the well-known explanation conveyed by (5), but also others in unlimited number, some of them very serviceable.

7. An outline of the Calculus of Enlargement has now been presented. Its brevity places it under a certain disadvantage, yet to treat the subject properly would require the preparation of a complete digest of the Calculus. Not having immediate opportunity to elaborate a work covering so much ground, I am compelled to confine myself for the present to a statement of the general principles on which such a digest should be prepared. The remainder of this essay will be devoted to the presentation of such new special theories as seem needed to complete the system.

B. Suggestions in Detail.

I. *Theory of Logarithms.*

8. An obvious objection to the use of $\log \mathrm{E}$ as the definition of D lies in the obscurity of the idea of the logarithm of an operative symbol; and to go further back, this obscurity is due to the difficulty of comprehending logarithms at all. It is said by De Morgan (*Calculus*, p. 126) that the only definition of $\log x$ used in analysis is y, where $\varepsilon^y = x$. When x and y are not numerical quantities, this is clearly unintelligible. It is certainly impossible to understand the expression ε^{D}, so frequently employed, if we suppose it to mean, as it must mean unless otherwise defined, the Dth power of the constant ε. Even when x and y are numbers, the definition is but indirect at the best. The alternative definitions which I have to suggest correspond identically with the explanations which will, further on, be given concerning $\mathrm{D} = \log \mathrm{E}$. We may consider $\log x$ to be y, where $x = 1 + y + \frac{1}{2} y^2 + \frac{1}{2\,.\,3} y^3 + \ldots$; or we may regard it as a vanishing fraction, or as an infinite series. The simplest series, and probably the most intelligible definition, is Mercator's well-known series,

$$\log (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots . \qquad (6)$$

9. Whatever definition of log x be adopted, it will be desirable to lay down the following definition of an antilogarithm. The series $1 + y + \frac{1}{2} y^2 + \frac{1}{2.3} y^3 + \ldots$ is a function of y; let it be known as the antilogarithm of y, and let it be denoted by the functional symbol ε^y. We may proceed as follows to investigate the properties of this symbol. By actual multiplication of the series, we shall find that $\varepsilon^x \varepsilon^y = \varepsilon^{x+y}$, where x and y may have any possible meaning. By an obvious extension of the same principle,

$$(\varepsilon^x)^h = \varepsilon^{xh}, \tag{7}$$

h being any numerical quantity, positive or negative. Putting $x = 1$, we see that $(\varepsilon^1)^h = \varepsilon^h$, from which we see that ε^h is equal to a certain constant raised to a power denoted by h. It is usual to call this constant ε. When h is not a symbol of quantity, it will be safe to regard ε^h as a symbol merely, according to its definition. In short, for all meanings of x, we have the well-known exponential theorem,

$$\varepsilon^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{2.3} x^3 + \ldots, \tag{8}$$

where, if x is a symbol of quantity, ε is a constant, whose value may be found by putting $x = 1$:

$$\varepsilon = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \ldots. \tag{9}$$

Having established this understanding concerning the symbol ε, we may define log x to be y, where $x = \varepsilon^y$, or where $x = 1 + y + \frac{1}{2} y^2 + \ldots$; and the various theorems concerning logarithms may be developed in the usual manner.

10. Another and, when duly weighed, most satisfactory definition may be derived from any one of an unlimited number of vanishing fractions, special cases of the general form

$$\log x = \frac{x^{(1-a)h} - x^{-ah}}{h}, \tag{10}$$

where h is infinitely reduced, that is to say, more briefly, where $h = 0$. This fraction is doubtless novel, though one case of it, where $a = 0$, is known. Even that case has not, I presume, been suggested heretofore as a definition. From (10) we have at once, substituting the equivalent series for ε^y,

$$\log \varepsilon^y = y. \tag{11}$$

The various theorems pertaining to logarithms may be derived with the

utmost facility by the aid of these vanishing-fraction definitions. Thus, if $a = 0$, we have, by expansion,

$$\log(\varepsilon^x \varepsilon^y) = \frac{(\varepsilon^x \varepsilon^y)^h - 1}{h}_{[h=0]} = x + y = \log \varepsilon^{x+y}, \tag{12}$$

$$\log(1+x) = \frac{(1+x)^h - 1}{h}_{[h=0]} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots. \tag{13}$$

11. Equation (6) furnishes, perhaps, the most intelligible definition of a logarithm. It is easy to form the idea of a function of the form $x - \frac{1}{2}x^2$, and the conception is not rendered more difficult by adding a term $\frac{1}{3}x^3$, or a multitude of terms similar in form. The notion of the sum of a series of integral powers is simpler than that of a vanishing fraction, and is also simpler than the customary notion of a logarithm, which involves, in an obscure and inverted manner, a fractional, or rather incommensurable, power of a strange looking constant. For instance,

$$\log \frac{3}{2} = \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \dots = 0.405 + \tag{14}$$

is a more intelligible definition than $\log \frac{3}{2} = y$, where $\frac{3}{2} = \varepsilon^y$, where $\varepsilon = 1 + 1 + \frac{1}{2} + \dots$. When x lies between 1 and -1, the series (6) is convergent, and the value of the logarithm may be obtained by approximation. When x is algebraically greater than 1, the series is divergent, but it may readily be shown that its sum is finite. Assuming what will shortly be proved, that if $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$, $x = y + \frac{1}{2}y^2 + \frac{1}{2 \cdot 3}y^3 + \dots$, one series being algebraically the reverse of the other, we observe that the latter series is essentially convergent, and that when $y = 0$, $x = 0$; when $y = \infty$, $x = \infty$; and when y varies continuously from 0 to ∞, x does the same, having a positive finite value for every positive finite value of y. The converse proposition is, therefore, true, that y, or $\log(1+x)$, is positive and finite for every positive finite value of x. The assumption just made is legitimate, for the proof of the reversion will certainly be accepted when $x < 1$, and the law of the coefficients of the reverse series cannot be different when x has any other value.

12. The various theorems relating to logarithms may easily be derived from this definition. Thus, by the binomial theorem, supposing $1 < a < 2$,

$$a^x = (1 + a - 1)^x = 1 + x(a-1) + x\frac{x-1}{2}(a-1)^2 + \dots, \tag{15}$$

which may be written

$$a^x = 1 + xc_1 + x^2c_2 + \ldots, \tag{16}$$

where $c_1 = \log a$. Hence,

$$a^{x+y} = a^y(1 + xc_1 + x^2c_2 + \ldots) = 1 + (x+y)\,c_1 + (x+y)^2\,c_2 + \ldots. \tag{17}$$

Placing the coefficients of x equal to each other, a proceeding to which in this case no objection can be urged, we have

$$a^yc_1 = c_1 + 2c_2y + 3c_3y^2 + \ldots. \tag{18}$$

But

$$a^yc_1 = c_1 + c_1c_1y + c_1c_2y^2 + \ldots; \tag{19}$$

hence $c_2 = \frac{1}{2}\,c_1^2$, $c_3 = \frac{1}{3}\,c_1c_2 = \frac{1}{2\,.\,3}\,c_1^3$, and so on, so that

$$a^y = 1 + y\log a + \frac{1}{2}\,y^2(\log a)^2 + \frac{1}{2\,.\,3}\,y^3(\log a)^3 + \ldots. \tag{20}$$

Since, by (7),

$$(\varepsilon^{\log a})^h = \varepsilon^{h\log a}, \tag{21}$$

we perceive, on comparison of (20) with (8), that, if $n = \log a$,

$$\varepsilon^{\log a} = a, \tag{22}$$

$$\varepsilon^{ny} = 1 + ny + \frac{1}{2}\,n^2y^2 + \ldots. \tag{23}$$

The applicability of (22) is limited by the supposition that $1 < a < 2$. This limitation may now be removed. Suppose $m = a - 1$, then $n = \log(1+m)$, and if from the series $\log(1+m)$ we seek by reversion to determine the value of m, we find it to be, however far the reversion may be carried,

$$m = n + \frac{1}{2}\,n^2 + \frac{1}{2\,.\,3}\,n^3 + \ldots. \tag{24}$$

We see by (22) that the law of this series is true for certain values of n, and the coefficients, independent of n, must be the same for all other values, so that (22) is universally true. Hence, for all meanings of x and y,

$$\varepsilon^{\log(xy)} = xy = \varepsilon^{\log x}\varepsilon^{\log y} = \varepsilon^{\log x + \log y}, \tag{25}$$

$$\log(xy) = \log x + \log y; \tag{26}$$

and again, from (21), h being a symbol of quantity, and u having any assignable meaning,

$$u^h = \varepsilon^{h\log u}, \tag{27}$$

$$\log u^h = h\log u. \tag{28}$$

13. That $\log x$ may be expressed in terms of x is well known. It is only necessary to write out the development of

$$\log x = \log\frac{1+x}{1+x^{-1}} = \log(1+x) - \log(1+x^{-1}). \tag{29}$$

It is possible that the fact has not been noticed that an unlimited number of similar developments may be produced, the general form being

$$\log x = \frac{1}{n}\left[\log(1+x^n) - \log(1+x^{-n})\right], \tag{30}$$

n having any value, positive or negative.

14. To explain the meaning of $\mathrm{D} = \log \mathrm{E}$, we must employ such expressions as can be found equivalent to $\log x$, substituting E for x; and it is desirable, to ensure breadth of view, to find as many such expressions as possible. I shall now present a *general logarithmic series*, which will be found to include as special cases not only two or three expressions already known, but also several important expressions hitherto unknown, besides an unlimited number of less useful variations. Let $y = x^{h(1-a)} - x^{-ha}$;* then

$$\log x = \frac{y}{h}\left(1 + \frac{2a-1}{2}y + \frac{3a-1}{2}\,\frac{3a-2}{3}y^2 + \ldots\right). \tag{31}$$

This series may be derived from (6) by writing, for $(1+x)$, $x^h = 1 + yx^{ha}$, and performing the necessary successive substitutions; but this process does not seem capable of furnishing a satisfactory algebraic demonstration. For the present, I must content myself with saying that the law of the series may be verified by reversion to any given extent, and that it may be demonstrated at once by Lagrange's theorem, as well as by another, and perhaps simpler, expansion theorem which will be given further on. The more important special cases are separately susceptible of algebraic proof, so that the temporary lack of a complete demonstration of the general series is not perceptibly detrimental, though certainly to be regretted.

15. Since a and h may have any value, the number of logarithmic formulæ which may be deduced from the general series is infinite. For h, however, but two values, 1 and 0, can advantageously be taken, all other values giving results substantially equivalent to those obtained when $h = 1$. Let us first consider the case where $h = 0$, and consequently $y = 0$. In this case all terms vanish except the first, which we may call the *general logarithmic vanishing fraction:*

$$\log x = \frac{y}{h} = \frac{x^{(1-a)0} - x^{-a0}}{0} = \frac{0}{0}. \tag{32}$$

We interpret this, of course, to mean that $\log x$ is the limit of the ratio of $x^{(1-a)h} - x^{-ah}$ and h, when h is indefinitely reduced. I shall have frequent

*Formulæ more symmetrical, though less simple, may be obtained by writing $\frac{1}{2}(1-b)$ for a.

occasion to use the symbol 0 to express a variable to which the value 0 is to be assigned, as in the present instance. Concerning vanishing fractions in general I shall have more to say later. A simple proof of (32) may be had by expanding, in terms of h, the ratio mentioned, employing the exponential theorem, and afterwards making $h=0$. In fact, a formula still more general in form may thus be obtained. For, u being any function of x having a finite logarithm,

$$u^h x^h = 1 + h\,(\log u + \log x) + h^2\text{P}\,, \textit{ suppose}; \tag{33}$$

$$u^h = 1 + h \log u + h^2\text{Q}\,. \tag{34}$$

Subtracting, dividing by h, and making $h=0$, we have the general formula in question, which, like (32), is probably new,

$$\log x = \frac{u^0 x^0 - u^0}{0}. \tag{35}$$

If, in (32), we put $a=0$, $a=1$, $a=\frac{1}{2}$, respectively, we have

$$\log x = \frac{x^0-1}{0}, \tag{36}$$

$$\log x = \frac{1-x^{-0}}{0}, \tag{37}$$

$$\log x = \frac{x^{\frac{0}{2}} - x^{-\frac{0}{2}}}{0}, \tag{38}$$

of which equations the first is known.

16. Before making $h=0$, let $a=-\frac{c}{h}$, where c is any arbitrary quantity, either positive or negative, so that $\frac{y}{h}=x^c\,\frac{x^h-1}{h}=x^c\log x$. Let $z=-ay$ $=\frac{cy}{h}=cx^c\log x$. Then, from (31),

$$\log x = \frac{1}{c}\left(z - z^2 + \frac{3}{2}\,z^3 - \frac{4^2}{2\,.\,3}\,z^4 + \frac{5^3}{2\,.\,3\,.\,4}\,z^5 - \ldots\right). \tag{39}$$

We may notice particularly two special cases. If $c=1$,

$$\log x = x \log x - (x\log x)^2 + \frac{3}{2}\,(x \log x)^3 - \ldots; \tag{40}$$

while if $c=-1$,

$$\log x = \frac{\log x}{x} + \left(\frac{\log x}{x}\right)^2 + \frac{3}{2}\left(\frac{\log x}{x}\right)^3 + \ldots. \tag{41}$$

These interesting series appear to be new. The first of the three is not in reality more general than the others, since it may be derived from the second

by writing x^c, and from the third by writing x^{-c}, for x. We may verify (40) to any desired extent by reversion of

$$x \log x = \varepsilon^{\log x} \log x = \log x + (\log x)^2 + \frac{1}{2} (\log x)^3 + \dots \quad (42)$$

17. If $h = 1$, we have the general series in the following simplified form, still probably novel,

$$\log x = y + \frac{2a-1}{2} y^2 + \frac{3a-1}{2} \frac{3a-2}{3} y^3 + \dots, \quad (43)$$

where $y = x^{1-a} - x^{-a}$. If $a = 0$, $y = x - 1$, and

$$\log x = (x-1) - \frac{1}{2} (x-1)^2 + \dots, \quad (44)$$

as by (6). If $a = 1$, $y = 1 - x^{-1}$, and

$$\log x = (1 - x^{-1}) + \frac{1}{2} (1 - x^{-1})^2 + \dots, \quad (45)$$

which expression, due, I believe, to Lagrange, may be regarded as conjugate to the one preceding. If $a = \frac{m}{n}$, a proper fraction, the coefficients of y^n, y^{2n}, &c., disappear.

18. If $h = 1$ and $a = \frac{1}{2}$, the resulting series is remarkable, since every alternate term disappears, and those terms which remain converge rapidly when x is not far from 1. Supposing $t = \frac{y}{2} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{2}$, the series is as follows:

$$\log x = 2\left(t - \frac{1}{2}\frac{t^3}{3} + \frac{1}{2}\frac{3}{4}\frac{t^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{t^7}{7} + \dots\right).^* \quad (46)$$

The law of the coefficients may be proved as follows. Let $u = \frac{x-1}{x+1}$; then $x = \frac{1+u}{1-u}$, and $\log x = \log (1+u) - \log (1-u)$. In the expansion of this expression let u be replaced by its equivalent $t(1+t^2)^{-\frac{1}{2}}$, and let the several powers of the binomial $1 + t^2$ be developed. It will be found that the coefficient of t^n, for even values of n, is 0; for odd values, let $m = \frac{1}{2} n$, and the coefficient of t^n will be composed of $m + \frac{1}{2}$ terms of the series

*A special case of this formula, giving $\log \varepsilon$ in terms of $\frac{\varepsilon^{\frac{1}{2}} - \varepsilon^{-\frac{1}{2}}}{2}$, has for some years been known, and it is surprising that its generalized application to all logarithms should not heretofore have been suggested. The formula for $\log \varepsilon$ was first published, so far as I am aware, in a communication made in 1865 by Hansen to the Royal Society of Saxony; but he did not assign the law of the series, which was communicated by Mr. T. B. Sprague in 1871 to Mr. W. M. Makeham, and published in the *Journal of the Institute of Actuaries.* Mr. Sprague's proof was by the method of indeterminate coefficients, with differentiation.

$\frac{1}{m}\left(1-m+m\frac{m-1}{2}-m\frac{m-1}{2}\frac{m-2}{3}+\ldots\right)$, the sum of which, by a known algebraic formula, is $\frac{(1-m)^{(m-\frac{1}{2}}}{m\,.\,1^{(m-\frac{1}{2}}}$, where $x^{(r}=x(x+1)(x+2)\ldots(x+r-1)$.

Thus, if $n=1$, the coefficient is $\frac{\left(\frac{1}{2}\right)^{(0}}{\frac{1}{2}\,.\,1^{(0}}=2$; if $n=3$, it is $\frac{\left(-\frac{1}{2}\right)^{(1}}{\frac{3}{2}\,.\,1^{(1}}=-\frac{1}{3}$ or $2\left(-\frac{1}{2}\frac{1}{3}\right)$, and so on, as in (46).

19. For substitution in (46), let $z=\frac{1}{x-1}$, so that $x=\frac{z+1}{z}$; also, let $u=4z(z+1)=t^{-2}$. Making these substitutions, and multiplying both members by $\sqrt{(z[z+1])}$, we have

$$\sqrt{(z[z+1])}\log\frac{z+1}{z}=1-\frac{1}{2}\frac{1}{3u}+\frac{1}{2}\frac{3}{4}\frac{1}{5u^2}-\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7u^3}+\ldots, \tag{47}$$

a formula which will be found useful in the computation of logarithms, and which may be compared with the known series,

$$\left(z+\frac{1}{2}\right)\log\frac{z+1}{z}=1+\frac{1}{3v}+\frac{1}{5v^2}+\ldots, \tag{48}$$

where $v=4\left(z+\frac{1}{2}\right)^2$. In the one, we have, for determining $\log\frac{z+1}{z}$, to make use of $\sqrt{(z[z+1])}$, the geometrical mean between z and $z+1$, while in the other we have to employ $z+\frac{1}{2}$, the arithmetical mean. Suppose that log 3, and therefore log 9, are known, and that it is desired to calculate log 10. Employing the usual formula (48), we have a very convergent series,

$$\frac{19}{2}\log\frac{10}{9}=1+\frac{1}{1083}+\frac{1}{651605}+\ldots; \tag{49}$$

but by (47) we obtain a series still more highly convergent,

$$\sqrt{90}\log\frac{10}{9}=1-\frac{1}{2160}+\frac{1}{1728000}-\ldots. \tag{50}$$

20. I conclude these suggestions concerning the theory of logarithms by presenting two novel approximative expressions. First,

$$\log x=\frac{x+1}{3}\frac{x^3-1}{x^3+x},\ \textit{nearly}, \tag{51}$$

whenever x is not far from 1. By development in terms of $x-1$, we find that this expression differs from $\log x$ by a quantity arithmetically less than

$\frac{(x-1)^5}{20}$. For example, if $x = 0.99$, $\log x = -0.010050335858$, nearly, a result too great by 4 in the 12th place of decimals. Again,

$$\log x = 2\,\frac{x-1}{3}\left(\frac{1}{x+1} + \sqrt{\frac{1}{x}}\right), \textit{ nearly}, \tag{52}$$

whenever x is not far from 1, the error being arithmetically less than $\frac{(x-1)^5}{320}$. For example, $\log 0.99 = -\frac{2}{597} - \frac{\sqrt{11}}{495}$, nearly, which is correct in the 12th place.

II. *General Theory of Operations.*

21. Algebra takes any symbols subject to these three laws,

$$x(y+z) = xy + xz, \tag{53}$$

the law of distribution;

$$xy = yx, \tag{54}$$

the law of commutation, and

$$x^m x^n = x^{m+n}, \tag{55}$$

the law of indices, and proves that certain theorems concerning such symbols follow necessarily from the laws. The various theorems of algebra are as true of all operative symbols subject to the three laws in question as they are of common symbols of quantity. Any correct process of reasoning applied to such symbols of operation produces correct results, by precisely that kind of proof which it is necessary to employ regarding symbols of quantity. There is no novelty in these preliminary statements. At first, the symbolic method was used as an instrument of discovery only with the utmost caution, and its results were not fully accepted until otherwise verified. Its absolute trustworthiness has, however, been established by the clearest methods of demonstration, and no mathematician now doubts the algebraic truth of any intelligible symbolic result. If any doubt remains, it is when a divergent series appears.

22. I would define a simple operation to be one which changes a function by alteration of the variable. For example, the change of ϕx (it would be more formal to write $\phi[x]$, but I shall omit the brackets where no ambiguity can arise) into $\phi\psi x$ is a simple operation. All simple operations are obviously

distributive, though some distributive operations are not simple. For example, $\mathrm{D}x^m = mx^{m-1}$ is not a simple operation.

23. Let s^h represent any operation such that

$$\mathrm{s}^h\phi x = \phi\psi^{-1}(\psi x + h)\,. \tag{56}$$

Here s indicates the kind of operation, depending on the form of ψ, and h represents the degree to which it is carried.

$$\begin{aligned}\mathrm{s}^m\mathrm{s}^n\phi x &= \mathrm{s}^m\phi\psi^{-1}(\psi x + n) = \phi\psi^{-1}(\psi\psi^{-1}[\psi x + n] + m)\\ &= \phi\psi^{-1}(\psi x + m + n) = \mathrm{s}^{m+n}\phi x\,.\end{aligned} \tag{57}$$

The operation s^h is, therefore, subject to the law of indices, and it will similarly be seen that s^m and s^n are commutative with each other and with constants, that is to say,

$$\mathrm{s}^m\mathrm{s}^n c\phi x = c\mathrm{s}^n\mathrm{s}^m\phi x\,. \tag{58}$$

For s^1 it will be sufficient to write s, without the index. The index h may, of course, have any value, whole or fractional, positive or negative, or it may even be a meaningless symbol; meaningless, that is, until some meaning is arbitrarily assigned to it.

24. Let $f_1\mathrm{s}$ be any function of s, the general form being

$$f_1\mathrm{s} = a_1\mathrm{s}^{p_1} + a_2\mathrm{s}^{p_2} + a_3\mathrm{s}^{p_3} + \ldots, \tag{59}$$

where $a_1, a_2 \ldots, p_1, p_2 \ldots$, are independent of s, and have any assignable meaning, so that

$$f_1\mathrm{s}\phi x = (a_1\mathrm{s}^{p_1} + \ldots)\,\phi x = a_1\phi\psi^{-1}(\psi x + p_1) + \ldots. \tag{60}$$

It will be seen, on examination, that all such functions are distributive and repetitive, and it is easy to show that they are also commutative. Let $f_2\mathrm{s}$ be another such function, say

$$f_2\mathrm{s} = b_1\mathrm{s}^{q_1} + b_2\mathrm{s}^{q_2} + \ldots; \tag{61}$$

then

$$f_1\mathrm{s}f_2\mathrm{s}\phi x = f_2\mathrm{s}f_1\mathrm{s}\phi x\,. \tag{62}$$

The general term of $f_1\mathrm{s}$ is, let us say, $a_m\mathrm{s}^{p_m}$, and that of $f_2\mathrm{s}$, $b_n\mathrm{s}^{q_n}$; then the general term of $f_1\mathrm{s}f_2\mathrm{s}$ will be $a_m\mathrm{s}^{p_m}b_n\mathrm{s}^{q_n}$, and that of $f_2\mathrm{s}f_1\mathrm{s}$ will be $b_n\mathrm{s}^{q_n}a_m\mathrm{s}^{p_m}$, which, by (58), are seen to be equivalent expressions. It follows that all terms of the two expansions correspond, so that the operations denoted by $f_1\mathrm{s}$ and $f_2\mathrm{s}$, that is to say, all functions of s, including constants, are commutative with each other. It follows that all functions of s may be combined or transformed in any usual algebraic manner, apart from the subject upon which they operate.

25. The consideration of the functions of S, after the form of ψ, and therefore that of S, have been assigned, constitutes, in the nomenclature of this essay, a Calculus. Since ψ, and therefore S, may have any form, there may be an infinite number of such branches of science called Calculus. The operations comprised under one calculus will not usually be commutative with those of another, but two or more operations belonging to different systems may be treated separately from the subject on which they are performed, provided care be taken not to change their order.

26. In every calculus the most important branch is that which corresponds to the theory of logarithms in algebra. There are several important theorems which are thus, in a sense, common to all such systems, having their common origin in the theory of logarithms. Whatever be the meaning of S, let $\text{R} = \log \text{S}$; then from paragraphs 13–18 we shall derive at once a number of expressions giving R in terms of S or of simple functions of S, expressions which it is not necessary, for present purposes, to write out. As an illustration, we have from (36)

$$\text{R} = \frac{\text{S}^0 - 1}{0}. \tag{63}$$

Let $\psi x = x^{-1}$, and let what S becomes under this supposition be denoted by H; then

$$\text{H}^h \phi x = \phi \frac{x}{1 + xh}, \tag{64}$$

and the calculus composed of all functions of the symbol H may be called the Calculus of H. To show the use of (63), let $\phi x = x^n$; then, if $\text{G} = \log \text{H}$,

$$\text{G} x^n = \frac{x^n (1 + xh)^{-n} - x^n}{h}, \tag{65}$$

where $h = 0$, whence after development, assuming the binomial theorem,

$$\text{G} x^n = - n x^{n+1}. \tag{66}$$

Again,

$$\text{G} \log x = \frac{\log x - \log (1 + xh) - \log x}{h}, \tag{67}$$

where $h = 0$, whence

$$\text{G} \log x = - x. \tag{68}$$

27. The widest generalization of Taylor's theorem which I have been able to discover is that which gives S^h in terms of $h\text{R}$. Since $\text{S}^h = \varepsilon^{h\text{R}}$, we have from (23)

$$\text{S}^h = 1 + h\text{R} + \frac{1}{2} h^2\text{R}^2 + \frac{1}{2.3} h^3\text{R}^3 + \dots. \tag{69}$$

As a single illustration of this theorem, let us, in the Calculus of H as before, expand $\text{H}^h \log x = \log x - \log(1 + xh)$:

$$\log x - \log(1 + xh) = \log x + h\text{G} \log x + \frac{1}{2} h^2\text{G}^2 \log x + \ldots$$

$$= \log x - hx + \frac{1}{2} h^2x^2 - \frac{1}{3} n\, x^3 + \ldots. \tag{70}$$

28. A connecting link between any calculus, say that of S, and any other, say that of S′, may be found as follows. From (56),

$$\text{S}^n\phi x = \phi\psi'^{-1}(\psi'x + \psi'\psi^{-1}[\psi x + n] - \psi'x) = \text{S}'^{(\text{S}^n - 1)\psi'x}\phi x. \tag{71}$$

From (69), writing t and t' for series containing n^2 as a factor, we derive this transformation of (71),

$$(1 + n\text{R} + t)\,\phi x = (1 + [\text{S}^n - 1]\,\psi'x\,.\,\text{R}' + t')\,\phi x = (1 + [n\text{R} + t]\,\psi'x\,.\,\text{R}' + t')\,\phi x; \tag{72}$$

whence, equating the coefficients of n,

$$\text{R}\phi x = \text{R}\psi'x\,.\,\text{R}'\phi x. \tag{73}$$

Let $\phi x = \psi x = \psi' x$; then $\text{R}' = \text{R}$, and

$$\text{R}\psi x = \text{R}\psi x\,.\,\text{R}\psi x, \tag{74}$$

whence, generally,

$$\text{R}\psi x = 1. \tag{75}$$

For example, in the Calculus of H,

$$\text{G}x^{-1} = 1, \tag{76}$$

as by (66). We may, indeed, derive (75) directly from (63).

29. If there is more than one independent variable, it is proper to write S_x, S_y, &c., the subscript letter denoting the variable with respect to which the operation is performed. Of two simple operations, S_x^m, S_y^n, performed successively on $\phi(x, y)$, it is a matter of indifference which comes first, the result in either case being $\phi(\psi^{-1}[\psi x + m],\ \psi^{-1}[\psi y + n])$; and it might readily be proved that all functions of two such independent operations are commutative.

30. If the operation S_yS_z be performed on a function of u and v, where u is a function of y and v a function of z, and if we then make y and z both equal to x, the result is the same as if we first make y and z equal to x, and then operate with S_x. The same remark applies to all powers, and, therefore, to all functions, of S_yS_z. If, instead of y, we write $x\,|\,u$, which may be interpreted "x varying only in u", and instead of z, $x\,|\,v$, "x varying only in v", the double operation $\text{S}_{x\,|\,u}\text{S}_{x\,|\,v}$ is the same as S_yS_z, and is equivalent to S_x. The symbol $\text{S}_{x\,|\,u}$ represents what may be called a partial operation, performed with

respect to x in u. If $u = x$, we shall have the symbol $\mathrm{s}_{x\,|\,x}$, which is needlessly cumbrous in appearance, and may advantageously be replaced by the abbreviated form $\mathrm{s}_{|\,x}$. In general, χ being any function,

$$\phi \mathrm{s}_x \chi\,(u, v, w \ldots) = \phi\,(\mathrm{s}_{x\,|\,u} \mathrm{s}_{x\,|\,v} \mathrm{s}_{x\,|\,w} \ldots)\,\chi\,(u, v, w \ldots)\,. \tag{77}$$

Also,

$$\phi \mathrm{s}_{x\,|\,u} \chi\,(u, v) = \phi\,(\mathrm{s}_x \mathrm{s}_{x\,|\,v}^{-1})\,\chi\,(u, v)\,. \tag{78}$$

Substituting ϕ log for ϕ,

$$\phi \mathrm{R}_x \chi\,(u, v, w \ldots) = \phi\,(\mathrm{R}_{x\,|\,u} + \mathrm{R}_{x\,|\,v} + \mathrm{R}_{x\,|\,w} + \ldots)\,\chi\,(u, v, w \ldots)\,, \tag{79}$$

$$\phi \mathrm{R}_{x\,|\,u} \chi\,(u, v) = \phi\,(\mathrm{R}_x - \mathrm{R}_{x\,|\,v})\,\chi\,(u, v)\,. \tag{80}$$

As special cases, among others,

$$\mathrm{R}_x^n \chi\,(u, v, w \ldots) = \mathrm{R}_{x\,|\,u} + \mathrm{R}_{x\,|\,v} + \mathrm{R}_{x\,|\,w} + \ldots)^n\,\chi\,(u, v, w \ldots)\,, \tag{81}$$

$$\mathrm{R}_x uv = v\mathrm{R}_x u + u\mathrm{R}_x v\,. \tag{82}$$

31. If u is a function of x, any other function of x is of course a function of u, and may be operated upon by any function of s_u; but functions of s_u and functions of s_x are not usually commutative. It may be shown that s_u is equivalent to s'_x, where s' depends on ψ', and where u, ψ, and ψ' are so related that, when two are given, the third is determined by the equation

$$\psi u = \psi' x\,. \tag{83}$$

Starting with this equation, we have, successively,

$$\chi\psi\psi^{-1}\,(\psi u + n) = \chi\psi'\psi'^{-1}\,(\psi' x + n)\,, \tag{84}$$

$$\mathrm{s}_u^n \chi\psi u = \mathrm{s}_x'^n \chi\psi' x\,; \tag{85}$$

whence, since $\psi u = \psi' x$,

$$\mathrm{s}_u^n = \mathrm{s}_x'^n, \tag{86}$$

$$\phi \mathrm{s}_u = \phi \mathrm{s}'_x\,. \tag{87}$$

Thus, from (73), writing v for ϕx,

$$\mathrm{R}_x v = \mathrm{R}_x \psi u \,.\, \mathrm{R}_u v\,. \tag{88}$$

32. The simplest Calculus is, of course, that in which $\psi x = x$, and $\mathrm{s}\phi x = \phi\,(x + 1)$. Here the operation s is that which I have called Enlargement, and is denoted by the symbol E. This calculus may, therefore, properly be called the Calculus of Enlargement. The most important function of E is $\log \mathrm{E} = \mathrm{D}$, which corresponds to R in the foregoing general discussion, whenever s is replaced by E.

33. In (73), let $\psi x = x$, and let us write ψ and R for ψ' and R'; then putting $v = \phi x$,

$$\mathrm{D} v = \mathrm{D}\psi x \,.\, \mathrm{R} v\,, \tag{89}$$

$$\mathrm{R}v = \frac{\mathrm{D}v}{\mathrm{D}\psi x}, \tag{90}$$

$$\mathrm{R}u = \frac{\mathrm{D}u}{\mathrm{D}\psi x} = \frac{\mathrm{D}u}{\mathrm{D}v}\,\mathrm{R}v, \tag{91}$$

where, if $u = x$,

$$\mathrm{R}x = \frac{\mathrm{D}x}{\mathrm{D}\psi x}. \tag{92}$$

Again, from (73),

$$\mathrm{R}\phi x = \mathrm{R}x \,.\, \mathrm{D}\phi x. \tag{93}$$

34. It follows from (87) that all the processes of any calculus, say that of S', may be expressed in the language of any one given calculus, say that of S, by means of suitable artifices. It is therefore unnecessary to discuss in detail more than one of these systems; and the preference must naturally be given to the simplest of all, the Calculus of Enlargement. As a mere matter of interest, however, I shall, before closing this essay, make some suggestions concerning another calculus, comprising those operations which are functions of M, where $\psi x = \log x$, and

$$\mathrm{M}^h \phi x = \phi\,(x\varepsilon^h). \tag{94}$$

This system may, in want of a better term, be called the Calculus of Multiplication.

III. *Theory of the Functions of* E.

35. The symbol E has sometimes been defined as ε^{D}, sometimes as $1 + \Delta$, and sometimes as representing an operation such that $\mathrm{E}\phi x = \phi\,(x+1)$. It has also sometimes been used to denote the operation which changes ϕx into $\phi\,(x+h)$. We cannot now accept a definition in terms of Δ or D, for a simple operation ought not to be defined in terms of one more complex, nor can we agree that E shall be dependent on any arbitrary quantity h; $\mathrm{E}\phi x$ must be $\phi\,(x+1)$ and nothing else. Yet if $\mathrm{E}\phi x = \phi\,(x+1)$ express the definition of E, it will require considerable labor to prove that in all cases $\mathrm{E}^h\phi x = \phi\,(x+h)$, and then only when h expresses some positive or negative quantity; and the argument will not be free from ambiguity, since, for example, it might be hard to prove that $\mathrm{E}^{\frac{1}{2}}\phi x$ cannot be its own opposite, namely, $-\phi\left(x + \frac{1}{2}\right)$. I find it better to define E^h, like S^h, as a compound symbol representing that simple operation which changes ϕx into $\phi\,(x+h)$, whatever be the meaning of h. In this light we must regard E, when without an index, as an abbreviated

form of E^1. Since $c\text{E}^0\phi x = c\phi x$, we observe that $c\text{E}^0 = c$ and $\text{E}^0 = 1$, and hence that any constant may be regarded as a function of E of the form $c\text{E}^0$.

36. It would have been sufficient to define E^h as that special case of S^h where $\psi x = x$, and it may be said at once that all which has been shown to be true of S and its functions is true of E and its functions. If any one shall hereafter deem it best, for teaching the rudiments of the Calculus of Enlargement, to omit all mention of other possible systems, based on simple operations other than E, he will find it sufficient to say concerning E what has been said above concerning S, substituting x for ψx. It is not now necessary to repeat concerning E what has been proved in regard to all repetitive simple operations, and I shall confine my attention to certain properties pertaining to all functions of E as such. While nearly all of these properties are now no doubt first exhibited in this light, it will be seen that some of them are already known, more or less explicitly, as properties pertaining to algebraic functions of D. Such propositions will, however, be found to have been generalized, the properties hitherto known concerning algebraic functions of D being now exhibited concerning all functions of E, and therefore concerning all functions of D. It will be remarked that the theorems about to be stated regarding functions of E are developed more easily than if they were to be proved as relating to functions of D; particularly when the comparative ease with which E and D may be defined is taken into consideration, such definition being an essential element in either case.

37. If the general term of ϕx is $a_n x^n$, that of $\phi\text{E}_x\psi\,(x+y)$ is $a_n\text{E}_x^n\psi\,(x+y) = a_n\psi\,(x+n+y)$, supposing x and y to be independent, and this for the same reason is also the general term of $\phi\text{E}_y\psi\,(x+y)$, so that all terms correspond, and

$$\phi\text{E}_x\psi\,(x+y) = \phi\text{E}_y\psi\,(x+y)\,. \tag{95}$$

The same may be shown for any number of variables. Also,

$$\phi\text{E}_x\psi\,(x-y) = \phi\,(\text{E}_y^{-1})\,\psi\,(x-y)\,. \tag{96}$$

38. If the general term of ϕx is $a_n x^n$, and that of ψx is $b_m x^m$, the general term of $\phi\text{E}_x c^{xy}\,\psi\,(c^x)$ is $a_n\text{E}_x^n c^{xy} b_m c^{xm} = a_n b_m \text{E}_x^n c^{x(y+m)} = a_n b_m c^{(x+n)(y+m)}$. Similarly, $\psi\text{E}_y c_{xy}\,\phi\,(c^y)$ and $c^{xy}\,\phi\,(c^y\text{E}_x)\,\psi\,(c^x)$ will be found to have this same general term, so that

$$\phi\text{E}_x c^{xy}\,\psi\,(c^x) = \psi\text{E}_y c^{xy}\,\phi\,(c^y), \tag{97}$$

$$\phi\text{E}_x c^{xy}\,\psi\,(c^x) = c^{xy}\,\phi\,(c^y\text{E}_x)\,\psi)c^x), \tag{98}$$

$$\psi\text{E}_y c^{xy}\,\phi(c^y) = c^{xy}\,\phi\,(c^y\text{E}_x)\,\psi\,(c^x). \tag{99}$$

In general, for any number of independent variables,

$$\begin{aligned}\phi \mathrm{E}_x \psi \mathrm{E}_y \ldots \chi \mathrm{E}_u \zeta \mathrm{E}_v c^{xy \ldots uvw} \xi (c^{xy \ldots uv}) &\\ = \phi \mathrm{E}_x \psi \mathrm{E}_y \ldots \chi \mathrm{E}_u \xi \mathrm{E}_w c^{xy \ldots uvw} \zeta (c^{xy \ldots uw}) &\\ = c^{xy \ldots uvw} \psi (c^{x \ldots uvw} \mathrm{E}_y) \ldots \zeta (c^{xy \ldots uw} \mathrm{E}_v) \xi (c^{xy \ldots uv} \mathrm{E}_w) \phi (c^{y \ldots uvw}), &\qquad (100)\end{aligned}$$

where ϕ, ψ, &c., are arbitrary functions. Again, similarly,

$$\phi (c^y \mathrm{E}_x^h) \psi (c^x) = \psi (c_x \mathrm{E}_y^h) \phi (c^y). \qquad (101)$$

Again, since $\mathrm{E}_x^n c = \mathrm{E}_x^n x^0 c = c$,

$$\phi \mathrm{E}_x c = \phi 1 c. \qquad (102)$$

Here c represents anything independent of x.

39. There are many special cases of the foregoing propositions which are themselves important general theorems. Some of these will now be mentioned. If in (95) and (96) we make $y = 0$, we shall have

$$\phi \mathrm{E}\, \psi x = \phi \mathrm{E}_0\, \psi (x + 0), \qquad (103)$$

$$\phi \mathrm{E}\, \psi x = \phi (\mathrm{E}_0^{-1})\, \psi (x - 0); \qquad (104)$$

and from the former of these, observing (102),

$$\phi \mathrm{E} x = x \phi 1 + \phi \mathrm{E}_0 0. \qquad (105)$$

Again,

$$\phi \mathrm{E} \sin x = \phi \mathrm{E}_0 \sin (x + 0) = \sin x \phi \mathrm{E}_0 \cos 0 + \cos x \phi \mathrm{E}_0 \sin 0, \qquad (106)$$

$$\phi \mathrm{E} \cos x = \cos x \phi \mathrm{E}_0 \cos 0 - \sin x \phi \mathrm{E}_0 \sin 0. \qquad (107)$$

It may be observed that since $\cos n = \cos (-n)$, $\mathrm{E}_0^n \cos 0 = \mathrm{E}_0^{-n} \cos 0$, and in general,

$$\phi \mathrm{E}_0 \cos 0 = \phi (\mathrm{E}_0^{-1}) \cos 0. \qquad (108)$$

If, in (97), we make $y = 1$,

$$\phi \mathrm{E} c^x \psi (c^x) = \psi (\mathrm{E}_1) c^{x1} \phi (c^1), \qquad (109)$$

and if $\psi (c^x) = 1$,

$$\phi \mathrm{E} c^x = c^x \phi c. \qquad (110)$$

If $y = 0$,

$$\phi \mathrm{E} \psi (c^x) = \psi \mathrm{E}_0 c^{x0} \phi (c^0), \qquad (111)$$

where, if $\phi \mathrm{E} = 1$,

$$\psi (c^x) = \psi \mathrm{E}_0 c^{x0}, \qquad (112)$$

which may be regarded as one form of Herschel's theorem. If, in (98), we write ψx for $\psi (c^x)$, and put $y = 1$, we shall have

$$\phi \mathrm{E} c^x \psi x = c^x \phi (c\mathrm{E}) \psi x. \qquad (113)$$

If, in (99), $x = 1$, we shall have, writing x for y,

$$\psi \mathrm{E} c^x \phi (c^x) = c^x \phi (c^x \mathrm{E}_1) \psi (c^1). \qquad (114)$$

Similarly, supposing $x = 0$ in (99), and writing x for y,

$$\psi \mathrm{E} \phi (c^x) = \phi (c^x \mathrm{E}_0) \psi (c^0). \qquad (115)$$

For example, let $\psi \mathrm{E} = \mathrm{E}^h$; then

$$\phi\,(c^{x+h}) = \phi\,(c^x \mathrm{E}_0)\, c^{h0}. \tag{116}$$

If, in (101), we put $y = 1$,

$$\phi\,(c\mathrm{E}^h)\ \psi\,(c^x) = \psi\,(c^x \mathrm{E}_1^h)\ \phi\,(c^1)\,, \tag{117}$$

while if $y = 0$,

$$\phi\,(\mathrm{E}^h)\,\psi\,(c^x) = \psi\,(c^x \mathrm{E}_0^h)\ \phi\,(c^0)\,. \tag{118}$$

In all these theorems, as well as in the more general ones from which they are derived, c may have any value. If we assign to it the value ε, we shall produce another series of theorems, for the most part less general in character, which it is not now necessary to write out in full.

40. If we have to do with two or more independent variables, we are at liberty to regard them as being themselves functions of a single supposed variable, which let us call l, the form of the functions being such that $x = gl + g'$, $y = kl + k'$, &c., where g, g', &c., are arbitrary constants; for independent variables may be viewed either as equicrescent quantities, in which case they must be functions, of the form mentioned, of some standard variable, or as quantities to which arbitrary values may be assigned, in which case, again, there is no difficulty in accepting the foregoing statement.* Since

$$\mathrm{E}_l \phi x = \phi\,(gl + g + g') = \phi\,(x + g) = \mathrm{E}_x^g \phi x\,, \tag{119}$$

E_l is a function of E_x, and all functions of E_l will be commutative with all functions of E_x. For E_l I shall hereafter use the symbol e, and for $\mathrm{E}_{l\,|\,x}$, $\mathrm{E}_{l\,|\,y}$, &c., the symbols e_x, e_y, &c.

IV. *Analytical Theory of Differentiation.*

41. Let $\mathrm{D} = \log \mathrm{E}$, and $d = \log e$. The former statement is new only as a definition, while the latter is, I suppose, novel in all respects.† Both D and d are functions of E, and have, therefore, all the properties which pertain to such functions in general. The operation denoted by D is Differentiation. That denoted by $d = \mathrm{D}_l$ is in reality the same operation, performed with

* To quote language used by Lagrange on another subject, "quoique dans les fonctions de deux variables que nous considérons ici, les deux variables soient censées indépendantes, . . . rien n'empêche cependant qu'on ne puisse regarder ces variables elles-mêmes comme des fonctions d'une autre variable quelconque, mais fonctions indéterminées et arbitraires." *Calcul des Fonctions*, ed. 1806, p. 334.

† That is to say, taking e as it has just been defined, namely, as equivalent to E_l, the symbol of enlargement performed with respect to an assumed variable l, where l is such that $x = gl + g'$. Nevertheless, on the one hand, it is already not unusual to say that a differential may be regarded as a differential coefficient taken with respect to an assumed variable; and on the other hand, it has been noticed by Arbogast (*Calcul des Dérivations*, p. 376) that, using our notation, $d = \log \mathrm{E}_x^g$, where g is any arbitrary constant. The present statement connects these two ideas, and indicates the form of the relation between l and x.

respect to an imagined variable. If it be desired in any case to make a verbal distinction between D and d, the operation denoted by d may be called "taking the differential"; but the word differentiation has been so long used in both senses, and the danger of misunderstanding is usually so slight, that such a verbal distinction will not often be required.

42. The resultant of the operation D is usually known by the term Differential Coefficient, though it is also sometimes called Derived Function or Derivative, and that of the operation d is known as a Differential. The term Derivative cannot be permanently satisfactory unless the word Derivation be substituted for Differentiation, a proposal which would not be listened to. It is on every account desirable that the operation and its resultant should have cognate names. The terms Derivative and Differential Coefficient are more or less objectionable, the one as recalling too strongly Lagrange's doctrine of Derived Functions, a theory not now in general use as an explanation of differentiation, the other as indicating a mere appendage to a differential; and the latter term is besides insufferably cumbrous. The word Differentiation, though introduced only in the present century into the language, is now firmly rooted. To express the resultant of this operation, and as a substitute for the phrase Differential Coefficient, I venture to coin the noun Differentiate. To this noun, as denoting that which has been differentiated, there seems to be no etymological objection, since it follows the analogy of such words as graduate, associate, duplicate, postulate, delegate, &c.

43. Just as a differential is in one sense a differentiate, since $d = \mathrm{D}_l$, so also in another sense may a differentiate be regarded as a differential, since, if we put $g = 1$, we have $\mathrm{D}_x = \log \mathrm{E}_x^g = \log e = d$. The differentiate is the simpler of the two, analytically, while the differential is frequently the more useful and intelligible for practical purposes. As both may be embraced in the same theory, there is no sufficient reason for excluding either from consideration. If the imagined variable l represents time, the differential is a differentiate with respect to time, and is known as a Fluxion. On the other hand, if, in $x = gl + g'$, we have g infinitesimal, the differential will also be infinitesimal, since $d = \log e = \log \mathrm{E}_x^g = g\mathrm{D}_x$.

44. From (69) we have Taylor's theorem,

$$\mathrm{E}^h = 1 + h\mathrm{D} + \frac{1}{2} h^2\mathrm{D}^2 + \frac{1}{2.3} h^3\mathrm{D}^3 + \ldots, \tag{120}$$

$$\phi(x+h) = \phi x + h\mathrm{D}\phi x + \frac{1}{2} h^2\mathrm{D}^2\phi x + \ldots. \tag{121}$$

Applied to $\phi 0$, (120) becomes Maclaurin's theorem; to x^m, the binomial theorem. The symbolic form (120) is always true, and the theorem itself (121) is therefore formally correct, though the resulting series is not always algebraically intelligible, and, even when intelligible, cannot, unless convergent, be verified arithmetically.* The following modification, possibly novel, will sometimes be found useful:

$$\phi(x+h)=\phi x+h\mathrm{D}\phi\left(x+\frac{1}{2}h\right)+\frac{h^3\mathrm{D}^3\varphi\left(x+\frac{1}{2}h\right)}{4.2.3}+\frac{h^5\mathrm{D}^5\varphi\left(x+\frac{1}{2}h\right)}{4^2.2.3.4.5}+\ldots \quad (122)$$

This is found by subtracting the development of $\phi\left(x-\frac{1}{2}h\right)$ from that of $\phi\left(x+\frac{1}{2}h\right)$ and then writing $x+\frac{1}{2}h$ for x; or, symbolically, from $\mathrm{E}^h=1+(\mathrm{E}^{\frac{1}{2}h}-\mathrm{E}^{-\frac{1}{2}h})\,\mathrm{E}^{\frac{1}{2}h}$. To extend Taylor's theorem to functions of two or more variables, we have only to develop $\mathrm{E}_x^h\mathrm{E}_y^k\ldots=\varepsilon^{h\mathrm{D}_x+k\mathrm{D}_y+\ldots}$.

45. It was shown in paragraph 12 that when $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots$, $x=y+\frac{1}{2}y^2+\frac{1}{2.3}y^3+\ldots$. A more direct proof of this reversion, which is a step in the demonstration of Taylor's theorem, is as follows. Log $[1+a+b\,(1+a)]$ is a certain series of powers of the expression $a+b\,(1+a)$. Expanding these powers, which are all positive and integral, by the binomial theorem, and separating the series forming the coefficient of $b^n\,(1+a)^n$, then expanding $(1+a)^n$ and multiplying the result by the coefficient just separated, and finally separating from the product the series forming the coefficient of a^mb^n, we find it to be, for all values of m and n greater than 0,

$$(-1)^{n-1}\frac{1}{m^{m)}}\left[(n-1)^{m-1)}-mn^{m-1)}\right.$$
$$\left.+\ldots(-1)^r\frac{m^{r)}}{r^{r)}}(n+r-1)^{m-1)}\ldots(-1)^m(n+m-1)^{m-1)}\right], \quad (123)$$

where $x^{k)}=x\,(x-1)\ldots(x-k+1)$. The series enclosed in brackets is, by a theorem in finite differences, equal to 0; hence all terms in a^mb^n, that is to say, all terms which contain both a and b, vanish. The terms remaining, which contain a alone and b alone, are respectively

$$a-\frac{1}{2}a^2+\frac{1}{3}a^3-\ldots=\log\,(1+a), \quad (124)$$

* The expansions of $\varepsilon^{-\frac{1}{h}}$, $\varepsilon^{-\frac{1}{h^2}}$, &c., apparently convergent and untrue, are really divergent and unintelligible, as may be seen on examination of those of $\varepsilon^{-\frac{1}{x+h}}$, &c., when x is very small. The coefficient of h^n in each of the former expansions contains a term of the form $\infty^n\frac{\varepsilon^{-v}v^n}{2.3\ldots n}$, where v is infinite, a form which becomes ∞^∞ when $n=v$.

$$b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \ldots = \log(1+b). \tag{125}$$

Hence,

$$\log[1+a+b(1+a)] = \log[(1+a)(1+b)] = \log(1+a) + \log(1+b). \tag{126}$$

If $y = \log(1+x) = x - \frac{1}{2}x^2 + \ldots$, let us assume, as the reverted series,

$$x = y + c_2 y^2 + c_3 y^3 + \ldots. \tag{127}$$

Similarly, if $v = \log(1+u)$,

$$u = v + c_2 v^2 + c_3 v^3 + \ldots. \tag{128}$$

Since, by (126), $y + v = \log[(1+x)(1+u)]$,

$$(1+x)(1+u) = 1 + (y+v) + c_2(y+v)^2 + c_3(y+v)^3 + \ldots. \tag{129}$$

But, from (127) and (128),

$$(1+x)(1+u) = (1 + y + c_2 y^2 + \ldots)(1 + v + c_2 v^2 + \ldots). \tag{130}$$

Equating the coefficients of v, and then comparing those of y, y^2, etc., we find that $2c_2 = 1$, $3c_3 = c_2$, $4c_4 = c_3$, and so on; whence $c_2 = \frac{1}{2}$, $c^3 = \frac{1}{2.3}$, and so on. Here, throughout, x and y are symbols devoid of meaning.

46. From (75),

$$\mathrm{D}x = 1. \tag{131}$$

It follows that dx, which is equal to $g\mathrm{D}x$, is equal to g; that $y = k$, &c., showing that dx, dy, &c., are arbitrary constants when x, y, &c., are independent variables.

47. From (79),

$$\phi \mathrm{D}_x \psi(u, v, w, \ldots) = \phi(\mathrm{D}_{x|u} + \mathrm{D}_{x|v} + \mathrm{D}_{x|w} + \ldots)\psi(u, v, w, \ldots), \tag{132}$$

a general theorem which, though I do not remember having seen it, may be already known. If $\phi\mathrm{D}_x = \mathrm{D}_x^n$, we derive the following theorem, substantially due to Arbogast,

$$\mathrm{D}_x^n \psi(u, v, w, \ldots) = (\mathrm{D}_{x|u} + \mathrm{D}_{x|v} + \ldots)^n \psi(u, v, w, \ldots), \tag{133}$$

of which the next, known as Leibnitz's theorem, is a special case:

$$\mathrm{D}_x^n uv = (\mathrm{D}_{x|u} + \mathrm{D}_{x|v})^n uv. \tag{134}$$

If $n = 1$,

$$\mathrm{D}_x uv = \mathrm{D}_{x|u} uv + \mathrm{D}_{x|v} uv = v\mathrm{D}_x u + u\mathrm{D}_x v. \tag{135}$$

48. From (80), similarly,

$$\phi \mathrm{D}_{x|u} \psi(u, v) = \phi(\mathrm{D}_x - \mathrm{D}_{x|v})\psi(u, v), \tag{136}$$

from which

$$\mathrm{D}_{x|u}{}^n \psi(u, v) = (\mathrm{D}_x - \mathrm{D}_{x|v})^n \psi(u, v), \tag{137}$$

of which the following, ascribed by Price to Hargreave, is a special case:

$$\mathrm{D}_{x|u}{}^n uv = (\mathrm{D}_x - \mathrm{D}_{x|v})^n uv. \tag{138}$$

49. From (88),

$$D_x v = D_x u \,.\, D_u v\,.\text{*} \tag{139}$$

If $v = x$,

$$D_x u \,.\, D_u x = Dx = 1\,, \tag{140}$$

whence

$$D_x u = \frac{1}{D_u x}\,. \tag{141}$$

Again, from (139) and (141),

$$D_x u = \frac{D_x v}{D_u v} = \frac{D_v u}{D_v x}\,. \tag{142}$$

If $v = l$, we have, since $d = D_l$,

$$D_x u = \frac{du}{dx}\,. \tag{143}$$

It follows that wherever D_x is written we may read $\frac{d}{dx}$, and *vice versa*; and from (141) we see that this is true whether x is or is not an independent variable. Again, from (139),

$$D_{x\,|\,u} = D_x u \,.\, D_{|\,u}\,, \tag{144}$$

one of which expressions may always be replaced by the other.

50. All results obtained in the language of differentiates may of course be expressed at once, *mutatis mutandis*, in that of differentials, and *vice versa.* In the case of partial differentiation, the student should be informed that he will frequently meet with a certain ambiguous form of expression, which may be illustrated by saying that he will find $D_{|\,x}^{2} D_{|\,y} u$ written $\frac{d^3 u}{dx^2 dy}$. Perhaps it would be well to avoid this ambiguity in future by writing, for example, $\frac{d_x^2 d_y u}{dx^2 dy}$, where d_x and d_y must be regarded as abbreviations of $D_{l\,|\,x}$ and $D_{l\,|\,y}$.

51. A large number of theorems relating to all functions of D may be derived at once from those already obtained concerning functions of E. Most of those which I now proceed to mention are known, though, as hitherto proved, they are known only for such forms of function as can be expressed in integral powers of D. In deriving these theorems from (95–102), it will be seen that changes are sometimes made in the form of expression, such as writing ϕ log for ϕ, ε^k for c, &c.

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*If I were writing an elementary treatise, I should introduce, at this point and elsewhere, the usual proofs and illustrations, together with others to be hereafter suggested. I have, for present convenience, deferred the consideration of such explanations of D as are afforded by vanishing fractions and series, but wish it to be understood that my separation of the "analytical" from the "explanatory theory of differentiation" is wholly arbitrary, and ought by no means to be imitated in any methodical treatise on the Calculus of Enlargement.

$$\phi \mathrm{D}_x \psi (x+y) = \phi \mathrm{D}_y \psi (x+y), \qquad (145)$$

$$\phi \mathrm{D}_x \psi (x-y) = \phi (-\mathrm{D}_y) \psi (x-y), \qquad (146)$$

$$\phi \mathrm{D}_x \varepsilon^{kxy} \psi (kx) = \psi \mathrm{D}_y \varepsilon^{kxy} \phi (ky), \qquad (147)$$

$$\phi \mathrm{D}_x \varepsilon^{mx} \psi x = \varepsilon^{mx} \phi (m + \mathrm{D}_x) \psi x, \qquad (148)$$

$$\phi \mathrm{D}_y \varepsilon^{kxy} \psi (ky) = \varepsilon^{kxy} \psi (ky + \mathrm{D}_x) \phi (kx), \qquad (149)$$

$$\begin{aligned} &\phi \mathrm{D}_x \psi \mathrm{D}_y \ldots \chi \mathrm{D}_u \zeta \mathrm{D}_v \varepsilon^{kxy \ldots uvw} \xi (kxy \ldots uv) \\ &\quad = \phi \mathrm{D}_x \psi \mathrm{D}_y \ldots \chi \mathrm{D}_u \xi \mathrm{D}_w \varepsilon^{kxy \ldots uvw} \zeta (kxy \ldots uw) \\ &\quad = \varepsilon^{kxy \ldots uvw} \psi (kx \ldots uvw + \mathrm{D}_y) \ldots \xi (kxy \ldots uv + \mathrm{D}_w) \phi (ky \ldots uvw), \qquad (150) \end{aligned}$$

$$\phi (ky + h\mathrm{D}_x) \psi (kx) = \psi (kx + h\mathrm{D}_y) \phi (ky), \qquad (151)$$

$$\phi \mathrm{D}_x c = \phi 0 c. \qquad (152)$$

In all of these the variables are supposed to be independent. From (103–118) we have the following, which, although mere special cases of those just given, are still of great importance as general theorems:

$$\phi \mathrm{D} \psi x = \phi \mathrm{D}_0 \psi (x + 0), \qquad (153)$$

$$\phi \mathrm{D} \psi x = \phi (-\mathrm{D}_0) \psi (x - 0), \qquad (154)$$

$$\phi \mathrm{D} x = \phi 0 x + \phi \mathrm{D}_0 0, \qquad (155)$$

$$\phi \mathrm{D} \sin x = \sin x \phi \mathrm{D} \cos 0 + \cos x \phi \mathrm{D} \sin 0, \qquad (156)$$

$$\phi \mathrm{D} \cos x = \cos x \phi \mathrm{D} \cos 0 - \sin x \phi \mathrm{D} \sin 0, \qquad (157)$$

$$\phi \mathrm{D} \cos 0 = \phi (-\mathrm{D}) \cos 0, \qquad (158)$$

$$\phi \mathrm{D} \varepsilon^{kx} \psi (kx) = \psi \mathrm{D}_1 \varepsilon^{kx1} \phi (k1), \qquad (159)$$

$$\phi \mathrm{D} \varepsilon^{kx} = \varepsilon^{kx} \phi k, \qquad (160)$$

$$\phi \mathrm{D} \psi (kx) = \psi \mathrm{D}_0 \varepsilon^{kx0} \phi (k0), \qquad (161)$$

$$\phi x = \phi \mathrm{D}_0 \varepsilon^{x0}, \qquad (162)$$

$$\phi \mathrm{D} \varepsilon^{kx} \psi (kx) = \varepsilon^{kx} \psi (kx + \mathrm{D}_1) \phi (k1), \qquad (163)$$

$$\phi \mathrm{D} \psi (kx) = \psi (kx + \mathrm{D}_0) \phi (k0), \qquad (164)$$

$$\phi (x + h) = \phi (x + \mathrm{D}_0) \varepsilon^{h0}, \qquad (165)$$

$$\phi (k + h\mathrm{D}) \psi (kx) = \psi (kx + h\mathrm{D}_1) \phi (k1), \qquad (166)$$

$$\phi (h\mathrm{D}) \psi (kx) = \psi (kx + h\mathrm{D}_0) \phi (k0). \qquad (167)$$

From these, putting $k = 1$,

$$\phi \mathrm{D} \varepsilon^{x} \psi x = \psi \mathrm{D}_1 \varepsilon^{x1} \phi 1, \qquad (168)$$

$$\phi \mathrm{D} \psi x = \psi \mathrm{D}_0 \varepsilon^{x0} \phi 0, \qquad (169)$$

$$\phi \mathrm{D} \varepsilon^{x} \psi x = \varepsilon^{x} \psi (x + \mathrm{D}_1) \phi 1, \qquad (170)$$

$$\phi \mathrm{D} \psi x = \psi (x + \mathrm{D}_0) \phi 0, \qquad (171)$$

$$\phi (1 + h\mathrm{D}) \psi x = \psi (x + h\mathrm{D}_1) \phi 1, \qquad (172)$$

$$\phi (h\mathrm{D}) \psi x = \psi (x + h\mathrm{D}_0) \phi 0. \qquad (173)$$

The possible difficulty of expressing ϕD in terms of E cannot be urged as an objection to the foregoing deductions. We know that ϕD can be expressed in terms of D; that each such power of D can be expressed in terms of Δ, and again that each such power of Δ can be expressed in terms of E. Inasmuch as we know that ϕD can be expressed in terms of E, it is unnecessary to inquire the exact form of the expression.

52. From (152), where c is anything independent of x,

$$\text{D}_x c = 0\,. \tag{174}$$

Inversely, operating on both sides of this equation by D_x^{-1},

$$\text{D}_x^{-1}0 = c\,. \tag{175}$$

Here is an operation which creates something out of nothing; and since we cannot tell what that something may be, the results of this operation, and of all other operations which have the same creative faculty, must be indeterminate. I presume that, in general, all functions of E which cannot be expressed in positive integral powers of Δ are productive of indeterminate results. If any such operation, say B, is performed on $\phi x = \phi x + 0$, it produces, in addition to what we may call the principal form of $\text{B}\phi x$, a complementary function of x, the coefficients of which may be assigned at will. In the case before us, we perceive that when the operation D^{-1}, called Integration, and usually represented by the sign $\int$, is performed, we must introduce a complementary constant before we can venture to interpret the result. It is unnecessary to say much in this essay regarding integration. We shall have occasion to use the well-known definite integral $\int_0^\infty \varepsilon^{-x}x^m dx = \Gamma(1+m)$, of which the fuller formal description is $\text{D}^{-1}\varepsilon^{-x}x^m{}_{[x=\infty]} - \text{D}^{-1}\varepsilon^{-x}x^m{}_{[x=0]}$. The sign $\int$, as commonly used, may be considered as equivalent to D_l^{-1}, since

$$\int \phi x dx = \text{D}^{-1}\phi x = dx\,.\,\text{D}_l^{-1}\phi x = \text{D}_l^{-1}\phi x dx\,. \tag{176}$$

53. From (160),

$$\text{D}\varepsilon^{kx} = \varepsilon^{kx}k\,, \tag{177}$$

$$\text{D}\varepsilon^{x} = \varepsilon^{x}. \tag{178}$$

If $x = \varepsilon^\theta$, $\text{D}_\theta x = \varepsilon^\theta = x$, the reciprocal of which is $\text{D}_x\theta$, or

$$\text{D} \log x = x^{-1}. \tag{179}$$

Hence,

$$\text{D}x^m = \text{D}_x\theta\,.\,\text{D}_\theta x^m = x^{-1}.\, m\varepsilon^{m\theta} = mx^{m-1}. \tag{180}$$

From (158),

$$\text{D} \cos 0 = -\text{D} \cos 0 = 0\,. \tag{181}$$

By Maclaurin's theorem,

$$\sin x = x\text{D} \sin 0 + x^2\text{Q}\,,\ \textit{suppose}; \tag{182}$$

whence, assuming $\mathrm{D} \sin 0$ finite, which can be proved from (157) when x is infinitesimal,

$$\frac{\sin 0}{0} = \mathrm{D} \sin 0, \tag{183}$$

and since, by trigonometry, $\frac{\sin 0}{0} = 1$,

$$\mathrm{D} \sin 0 = 1. \tag{184}$$

Then, from (156) and (157),

$$\mathrm{D} \sin x = \cos x, \tag{185}$$

$$\mathrm{D} \cos x = -\sin x. \tag{186}$$

54. The following series, the result of integration by parts, a method deducible, as usual, from (135), are well known:*

$$\Gamma(1+p) = \int_0^x \varepsilon^{-x} x^p dx + \int_x^\infty \varepsilon^{-x} x^p dx$$

$$= \varepsilon^{-x} x^p \left[1 + \frac{x}{p+1} + \frac{x^2}{(p+1)(p+2)} + \ldots + p x^{-1} + p(p-1) x^{-2} + \ldots \right]; \tag{187}$$

$$\varepsilon^x = \left[1 + \frac{x}{p+1} + \ldots + p x^{-1} + \ldots \right] \frac{x^p}{\Gamma(1+p)}. \tag{188}$$

Let us write $h\mathrm{D}$ for x, and its antilogarithm E^h for ε^x, and let us suppose the subject of operation to be ϕx. We shall then obtain the following *extension of Taylor's theorem:*

$$\begin{aligned} \phi(x+h) = \Big[1 + \frac{h\mathrm{D}}{p+1} + \frac{h^2\mathrm{D}^2}{(p+1)(p+2)} + \ldots + p h^{-1} \mathrm{D}^{-1} \\ + p(p-1) h^{-2} \mathrm{D}^{-2} + \ldots \Big] \frac{h^p \mathrm{D}^p}{\Gamma(1+p)} \phi x. \end{aligned} \tag{189}$$

The original series terminates, in one direction, when p is an integer, so that in that case our extended theorem takes the usual form of Taylor's theorem. It will be observed that p may be any quantity except a negative integer. If $\phi x = x^m$, we shall have, as a special case, the extended binomial theorem of Roberts.† If we write 0 for x and x for h, we shall derive the following *extension of Maclaurin's theorem:*

$$\phi x = \left[1 + \frac{x\mathrm{D}_0}{p+1} + \ldots + p x^{-1} \mathrm{D}_0^{-1} + \ldots \right] \frac{x^p \mathrm{D}_0^p}{\Gamma(1+p)} \phi 0. \tag{190}$$

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* De Morgan, *Calculus*, p. 590; Roberts, *Quarterly Journal*, VII, p. 207.

† In this, as in (189), when p is an integer, there can be no powers of h with negative indices. Unaware of this limitation, Roberts obtained anomalous and perplexing results. How near he came to formulating the extension of Taylor's theorem may be seen from the following quotations: "Let $\phi(x)$ be any function of x, then $\frac{\mathrm{D}_x^n}{\Gamma(1+n)} \phi(x) =$ coefficient of a development of $\phi(x+h) \ldots [k']$, according to powers of h to the base under n." "All that the equivalence $[k']$ means is this: if $\phi(x+h)$ can be developed according to powers of $(x+h)$, $\frac{\mathrm{D}_x^n}{\Gamma(1+n)} \phi(x)$ [$\phi(x)$ being similarly developed] will give the corresponding coefficient of h^n in a development to the base index n."

While there may not now appear any practical use to which these theorems for expansion in fractional powers can be put, they will at least be found to throw some light on the theory of the subject. Various interesting series, such as for sin x, cos x, and of course ε^x, may be obtained by the use of (190), and x^m can be treated by it, when m is fractional, whereas Maclaurin's theorem cannot be employed in that case; the result of such treatment being that all terms except x^m vanish.

55. By employing (188) to expand ε^{x0} in (162), we have at once this *extension of Herschel's theorem*,

$$\phi x = \phi \mathrm{D}_0 \left[1 + \frac{x0}{p+1} + \ldots + px^{-1}0^{-1} + \ldots\right] \frac{x^p 0^p}{\Gamma(1+p)}. \tag{191}$$

Here, as before, p cannot be a negative integer.

56. I shall give more space to the consideration of the form of $\mathrm{D}^n x^m$, where n is fractional, than would be necessary were it not for the fact that it has been the subject of a noted controversy. Messrs. Liouville, Kelland and others make

$$\mathrm{D}^n x^m = (-1)^n \frac{\Gamma(-m+n)}{\Gamma(-m)} x^{m-n}, \tag{192}$$

while Peacock makes

$$\mathrm{D}^n x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-n)} x^{m-n}. \tag{193}$$

De Morgan (*Calculus*, p. 599) conjectures that "neither system has any claim to be considered as giving *the* form of $\mathrm{D}^n x^m$, though either may be *a* form." Later, Roberts shows, by strong arguments of analogy, that Peacock's form is tenable, while he admits the force of the arguments adduced in favor of that of Liouville. The reader cannot probably find in existence a more complete illustration of the difficulty with which such a subject is handled, under the indirect theory of differentiation heretofore followed, than that furnished by Roberts' argument. It is not too much to say that under that theory the meaning of D^n, where n is fractional, can only be guessed at. That indirect theory gives us D, the special case, and permits us to divine, if we can, by induction, analogy, or conjecture, the meaning of D^n, the general form. This is in every science the natural order of things so long as the general law, which shall furnish direct deductive proof, is unknown. The method now presented enables us to treat this case, like all others, with confidence and certainty. It makes us acquainted with D^n as one of many functions of E, and enables us to discuss, if we please, the general form D^n before the special form

D. If, in pursuance of the direct method, we arrive in any case at results which are not intelligible, we can only seek further for such expressions as we can understand, knowing that when found we can depend upon their accuracy, provided due allowance be made for possible complementary functions. I shall now try not only to show that Peacock's form is the principal form of $\mathrm{D}^n x^m$, but also to indicate the precise nature of the error made by his antagonists.

57. If, in (189), $\phi x = x^p$, we derive a development of $(x+h)^p$ in powers of h which, when p is a fraction, extends to infinity in both directions. If $\mathrm{D}^p x^p$ is a constant, the coefficients of h^{p+1}, h^{p+2}, &c., which are derived from $\mathrm{D}^p x^p$ by differentiation, will vanish. That $\mathrm{D}^p x^p$ is a constant may be shown from (173), which gives, writing z for h,

$$\mathrm{D}^p x^p = z^{-p}\,(x + z\mathrm{D}_0)^p 0^p, \tag{194}$$

wherein putting $z = x$ eliminates x. Omitting the vanishing terms of the development of $(x+h)^p$, and comparing the coefficient of h^p in the remainder of the development, namely, $\dfrac{\mathrm{D}^p x^p}{\Gamma(1+p)}$, with that of h^p in the known development of $(x+h)^p$ by the binomial theorem, namely, 1, we have

$$\mathrm{D}^p x^p = \Gamma\,(1+p)\,. \tag{195}$$

This equation is thus shown to be true for all cases except when p is a negative integer. That it is formally true in that case also may be seen upon repeated integration, resulting in a term containing $\mathrm{D}^{-1}x^{-1} = \log x = \dfrac{x^0}{0} - \dfrac{1}{0}$, the latter fraction being the complementary constant. Therefore, in all cases

$$\frac{\mathrm{D}^m x^m}{\Gamma(1+m)} = \frac{\mathrm{D}^{m-n} x^{m-n}}{\Gamma(1+m-n)}. \tag{196}$$

Operating on both sides with D^{n-m}, and multiplying by $\Gamma\,(1+m)$, we have, for all values of n, Peacock's formula,

$$\mathrm{D}^n x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-n)}\,x^{m-n}. \tag{197}$$

It may readily be shown that in this case there are no complementary terms in x^{m-n-1}, x^{m-n-2}, &c., such as might be created from 0 by the operation D^{n-m}. For, by (189), the coefficient of h^n in $(x+h)^m$ is $\dfrac{\mathrm{D}^n x^m}{\Gamma(1+n)}$, while we know, by the common expansion of $(x+h)^n(x+h)^{m-n} = (h^n + \ldots)(x^{m-n} + \ldots)$, that this coefficient contains no other power of x than x^{m-n}.

58. The arguments by which it has been proved that Liouville's form of $D^n x^m$ is correct have never been impugned, nor do I now impugn them, though I hold that it is not the principal form. If the performance of an ambiguous operation (see paragraph 52) such as D^n, where n is fractional, produces in one way a real result and in another an imaginary result differing from the real by a complementary function which D^n produces from 0, and which the inverse operation D^{-n} will reduce to 0, we are bound to accept the real result as the principal form. The proof of Liouville's formula depends on this equation, derived from (160),

$$D^n \varepsilon^{-xv} = (-v)^n \varepsilon^{-xv}. \tag{198}$$

When n is fractional, this expression is imaginary. It is, however, formally correct, and no one seems to have suspected that another and real expression can be found. Even Roberts explicitly lays down $D^n \varepsilon^{ax} = a^n \varepsilon^{ax}$, without limitation, as if it were the principal, or indeed the only possible, form. I shall now show that (198) is not the principal form of $D^n \varepsilon^{-xv}$. Since $\varepsilon^{-xv} = x^0 v^0 - xv + \frac{x^2 v^2}{2} - \dots$, we have, by (197),

$$D^n \varepsilon^{-xv} = \frac{x^{-n}}{\Gamma(1-n)} - \frac{x^{1-n} v}{\Gamma(2-n)} + \frac{x^{2-n} v^2}{\Gamma(3-n)} - \dots, \tag{199}$$

a series not only not imaginary, but also essentially convergent, and, therefore, eminently acceptable. If, on the other hand, we expand $(-v^n)\varepsilon^{-xv}$ by (188), writing $-xv$ for x, and $-n$ for p, we shall have the same series, with these additional terms, $-\frac{x^{-n-1}v^{-1}}{\Gamma(-n)} + \frac{x^{-n-2}v^{-2}}{\Gamma(-n-1)} - \dots$. Now the additional terms become ultimately all of the same sign, forming a series infinite in value, as might have been expected from the imaginary character of the function developed; but it is especially to be remarked that they all vanish when operated upon by D^{-n}, showing that they constitute a complementary function, and are not necessarily part of the principal form of $D^n \varepsilon^{-xv}$. Here then are two forms of $D^n \varepsilon^{-xv}$, (198) and (199), one imaginary, the other real, the former being composed of the real form plus a complementary function. The real form is therefore the principal one. It is, however, only by employing the imaginary form that the expression given for $D^n x^m$ by Liouville can be proved.

59. A good illustration of the ease with which secondary forms of such expressions as $D^n x^m$ may be obtained consists in the application of (171) to $D^n x^m$, whence

$$D^n x^m = (x + D_0)^m \, 0^n. \tag{200}$$

Now this binomial may be so expanded by Roberts' theorem as to produce a result differing from (197) by only a complementary function; but if on the other hand it is expanded in the usual way, in positive integral powers of D_0, it produces an expression, probably new,

$$\mathrm{D}^n x^m = (x^m + m x^{m-1}\mathrm{D}_0 + \ldots)\, 0^n, \tag{201}$$

which, although formally correct, can have no claim to be considered the principal form of $\mathrm{D}^n x^m$. It does, however, give correct real results when n is a positive integer, and in every case satisfies, as does Liouville's expression, the requirement of interpolation of form. For example,

$$\mathrm{D}^{\frac{1}{3}} x = x 0^{\frac{1}{3}} + \frac{1}{3}\, 0^{-\frac{2}{3}} x^0, \tag{202}$$

and repeating,

$$\mathrm{D}^{\frac{1}{3}}\mathrm{D}^{\frac{1}{3}} x = x 0^{\frac{2}{3}} + \frac{2}{3}\, 0^{-\frac{1}{3}} x^0, \tag{203}$$

$$\mathrm{D}^{\frac{1}{3}}\mathrm{D}^{\frac{1}{3}}\mathrm{D}^{\frac{1}{3}} x = \mathrm{D}x = x0 + \frac{3}{3}\, 0^0 x^0 = 1\,. \tag{204}$$

V. *Explanatory Theory of Differentiation.*

60. Although the various theorems of the Differential and Integral Calculus may readily be derived from the propositions already laid down, we have really taken but a narrow view of the subject. We have not done much more than to exhibit Taylor's theorem, and to ascribe to D as a function of E certain properties pertaining to such functions in general. We have now to examine more closely into the nature of the operation of differentiation, as disclosed by its symbolic definition, $\mathrm{D} = \log \mathrm{E}$.*

61. The coefficient of h in the expansion of $\phi\,(x+h)$ by Taylor's theorem is $\mathrm{D}\phi x$. If, therefore, we know the development of any function of $x+h$ in positive integral powers of h, we know at once the differentiate of the same function of x. Thus, from the binomial theorem, we have $\mathrm{D}x^m = mx^{m-1}$; from the exponential theorem, $\mathrm{D}\epsilon^x = \epsilon^x$; from the logarithmic series, $\mathrm{D}\log x = x^{-1}$; and from the trigonometrical series, $\mathrm{D}\sin x = \cos x$ and $\mathrm{D}\cos x = -\sin x$. This method of determining $\mathrm{D}\phi x$ rests on surer grounds than the somewhat

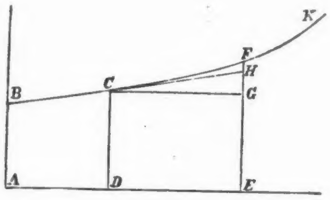
* I must again observe that the order in which, for present convenience, these several matters are discussed is not that which should be followed in a methodical treatise on the Calculus of Enlargement. In such a work, the elementary explanations which we have now to consider should be introduced as soon as practicable after the first mention of differentiation, and be followed up at every convenient point by suitable illustrations.

similar principle underlying the *Calcul des Fonctions* of Lagrange, for we have what he had not, a symbolic demonstration of Taylor's theorem; and, not to dwell too long upon it, we may pass it by with the remark that in all probability no insuperable objection can be made to it.

62. Perhaps it has not been noticed hitherto that a simple variation of Taylor's theorem,

$$\Delta = \text{D} + \frac{1}{2}\,\text{D}^2 + \frac{1}{2\,.\,3}\,\text{D}^3 + \ldots, \tag{205}$$

is remarkably susceptible of geometric illustration. For example, let $\text{AD} = x$, and let the space ABCD, included between the straight line AD, the curve BCK, and the two perpendiculars AB and DC, be called ϕx. Take $\text{DE} = 1$, and draw the lines EF perpendicular, and CG parallel, respectively, to AE; also CH tangent to the curve. Then,



$$\text{D}\phi x = \text{DEGC}, \tag{206}$$

$$\frac{1}{2}\,\text{D}^2\phi x = \text{CGH}, \tag{207}$$

$$\frac{1}{2\,.\,3}\,\text{D}^3\phi x + \ldots = \text{CHF}, \tag{208}$$

$$\Delta\phi x = \text{CDEF} = \text{D}\phi x + \frac{1}{2}\,\text{D}^2 x + \frac{1}{2\,.\,3}\,\text{D}^3 x + \ldots. \tag{209}$$

63. It is desirable to find as many expressions as possible for $\log x$ in terms either of x or of simple functions of x, and in them to write E for x, in order to arrive at the clearest understanding of the operation $\text{D} = \log \text{E}$ by attentive observation of its various algebraic equivalents. For this purpose the two general series (30, 31) presented in the foregoing Theory of Logarithms afford ample means.

64. From (30),

$$\text{D} = \frac{1}{n}\,\text{E}^n - \frac{1}{2n}\,\text{E}^{2n} + \ldots - \frac{1}{n}\,\text{E}^{-n} + \frac{1}{2n}\,\text{E}^{-2n} - \ldots, \tag{210}$$

where n may have any value. If $n = 1$,

$$\text{D} = \text{E} - \frac{1}{2}\,\text{E}^2 + \ldots - \text{E}^{-1} + \frac{1}{2}\,\text{E}^{-2} - \ldots. \tag{211}$$

Applied to ϕx,

$$\text{D}\phi x = \frac{1}{n}\,[\phi\,(x+n) - \phi\,(x-n)] - \frac{1}{2n}\,[\phi\,(x+2n) - \phi\,(x-2n)] + \ldots, \tag{212}$$

$$\mathrm{D}\phi x = [\phi(x+1) - \phi(x-1)] - \frac{1}{2}[\phi(x+2) - \phi(x-2)] + \ldots . \tag{213}$$

These series, which are probably new, will, owing to their symmetry of form, be readily borne in mind. To illustrate their use, let $\phi x = a^x$, and we have

$$\mathrm{D}a^x = a^x\left[\frac{1}{n}(a^n - a^{-n}) - \frac{1}{2n}(a^{2n} - a^{-2n}) + \ldots\right] = a^x \log a. \tag{214}$$

Let $\phi x = x^m$, then

$$\mathrm{D}x^m = [(x+1)^m - (x-1)^m] - \frac{1}{2}[(x+2)^m - (x-2)^m] + \ldots . \tag{215}$$

Here all terms in x^m, x^{m-2}, x^{m-4}, &c., obviously vanish. The terms in x^{m-3}, x^{m-5}, &c., contain, in the coefficient of each such power, a factor of the form $1 - 2^r + 3^r - \ldots$, where r is an even positive integer, so that, by a known theorem in finite differences, these terms likewise vanish. There remain the terms in x^{m-1}, whose coefficients are $2m - 2m + 2m - \ldots = m$, so that, finally,

$$\mathrm{D}x^m = mx^{m-1}. \tag{216}$$

Again, putting $n = \frac{\pi}{2}$,

$$\mathrm{D}\sin x = \frac{2}{\pi}\left[\sin\left(x + \frac{\pi}{2}\right) - \sin\left(x - \frac{\pi}{2}\right) - \ldots\right]$$

$$= \frac{2}{\pi}\left(2 - \frac{2}{3} + \frac{2}{5} - \ldots\right)\cos x = \cos x. \tag{217}$$

It is needless, however, to multiply illustrations which will readily occur to the reader.

65. Let the symbol ∂ represent the operation of obtaining the ratio of the most general form of difference of a function to the corresponding difference of its variable; that is to say, let

$$\partial = \frac{\mathrm{E}^{-ha+h} - \mathrm{E}^{-ha}}{h},* \tag{218}$$

$$\partial\phi x = \frac{\varphi(x - ha + h) - \varphi(x - ha)}{h}. \tag{219}$$

The constants a and h may have any value, so that there will be an unlimited number of special cases, some of which will, from their greater importance, require distinct symbols. Thus, when $h = 1$ and $a = 0$, $\partial = \mathrm{E} - 1 = \Delta$; when $h = 1$ and $a = 1$, $\partial = 1 - \mathrm{E}^{-1}$, which is sometimes denoted by $'\Delta$; when $h = 1$ and $a = \frac{1}{2}$, $\partial = \mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}$, which let us represent by Λ; and

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* Results more symmetrical, though less simple, can be got by writing $\frac{1}{2}(1 - b)$ for a.

when $h=0$ and $a=0$, $\partial = \text{D}$. It will shortly be shown that we need not restrict D to the case where $a=0$, but that $\partial = \text{D}$ when $h=0$, whatever be the (finite) value of a.

66. We now derive at once from (31) the following general *differentiate-expression:*

$$\text{D} = \partial + \frac{2a-1}{2} h\partial^2 + \frac{3a-1}{2}\frac{3a-2}{3} h^2\partial^3 + \frac{4a-1}{2}\frac{4a-2}{3}\frac{4a-3}{4} h^3\partial^4 + \dots \quad (220)$$

Of all special cases of this theorem, those are particularly important in which $h=0$ or $h=1$. When $h=0$, there are four chief cases, where $a=0$, $a=1$, $a=\frac{1}{2}$, and $a=\infty$, respectively; and when $h=1$, there are three chief cases, where $a=0$, $a=1$, and $a=\frac{1}{2}$, respectively. I shall discuss these in order.

67. Let $h=0$. In this case the general theorem is reduced to a vanishing fraction. Concerning vanishing fractions in general, it may be said that they are rendered needlessly obscure by presentation in the form $\frac{0}{0}$. Whenever we have to write 0 as the denominator of a fraction we ought, I think, if convenient, to express the numerator as a function of the denominator, or, in other words, as a function of 0, that symbol, when employed in the numerator, representing the denominator and nothing else. So expressed, it is impossible for a vanishing fraction to be ambiguous in meaning, supposing it possible to expand the numerator in positive integral powers of 0. It matters little whether such fractions are philosophically explained by the doctrine of infinitesimals or by that of limits. All that is necessary to their acceptance is to persuade ourselves in some way that $\frac{x}{x}=1$ when $x=0$.

68. When $h=0$, we have, therefore,

$$\text{D} = \frac{\text{E}^{(1-a)0} - \text{E}^{-a0}}{0}, \quad (221)$$

an expression which may be instantly verified by expansion. It may, indeed, be shown that

$$\text{D} = \frac{\text{P}^0(\text{E}^0 - 1)}{0}, \quad (222)$$

where P is any function of E. Of this (221) is a special case. In practice, we may apply (221) thus,

$$\text{D}\phi x = \frac{\varphi(x-ha+h) - \varphi(x-ha)}{h}_{[h=0]}. \quad (223)$$

The differentiate of any function, therefore, is equal to an infinitely small difference of the function divided by the corresponding difference of the variable; or, in other words, to the limit of the ratio of differences indefinitely reduced. In this statement it will be observed that the word Difference cannot be replaced by the word Increment without obscuring the truth which is conveyed. To illustrate (223), let $a = -4$; then

$$\mathrm{D}x^3 = \frac{(x+5h)^3 - (x+4h)^3}{h}_{[h=0]} = 3x^2, \tag{224}$$

$$\mathrm{D}\varepsilon^x = \frac{\varepsilon^{x+5h} - \varepsilon^{x+4h}}{h}_{[h=0]} = \varepsilon^x. \tag{225}$$

69. Since $\mathrm{D} = \dfrac{d}{dx}$, where dx is an arbitrary constant, let $dx = h$; then, when h is infinitesimal,

$$d = \mathrm{E}^{(1-a)0} - \mathrm{E}^{-a0}. \tag{226}$$

This is to be interpreted as an order to perform the operation $\mathrm{E}^{(1-a)h} - \mathrm{E}^{-ah}$, to make $h = 0$, and to represent the 0 in question by the symbol dx. Applied to ϕx, it becomes

$$d\phi x = \phi\,(x - adx + dx) - \phi\,(x - adx)\,. \tag{227}$$

When, again, instead of being infinitesimal, dx is taken to have some tangible finite value, dx and $d\phi x$ have nevertheless the same ratio as if both were infinitely small, so that when dx is assigned, and the ratio ascertained, the value of $d\phi x$ is known. The doctrine of fluxions is a case in point.

70. If, in (221), $a = 0$,

$$\mathrm{D} = \frac{\mathrm{E}^0 - 1}{0}. \tag{228}$$

This is the symbolic embodiment, possibly not new, of the usual expression

$$\mathrm{D}\phi x = \frac{\varphi\,(x+h) - \varphi x}{h}_{[h=0]}. \tag{229}$$

If $a = 1$,

$$\mathrm{D} = \frac{1 - \mathrm{E}^{-0}}{0}, \tag{230}$$

$$\mathrm{D}\phi x = \frac{\varphi x - \varphi\,(x-h)}{h}_{[h=0]}, \tag{231}$$

the latter again being a known form. If $a = \frac{1}{2}$,

$$\mathrm{D} = \frac{\mathrm{E}^{\frac{0}{2}} - \mathrm{E}^{-\frac{0}{2}}}{0}, \tag{232}$$

$$\mathrm{D}\phi x = \frac{\varphi\left(x + \frac{1}{2}h\right) - \varphi\left(x - \frac{1}{2}h\right)}{h}_{[h=0]}, \tag{233}$$

both of which expressions are probably new. The three forms thus derived by making $a=0$, $a=1$, and $a=\frac{1}{2}$, may be called the upper, lower, and central vanishing fractions respectively. Correspondingly, from (226) and (227),

$$d=\text{E}^0-1, \tag{234}$$
$$d\phi x=\phi\,(x+dx)-\phi x; \tag{235}$$
$$d=1-\text{E}^{-0}, \tag{236}$$
$$d\phi x=\phi x-\phi\,(x-dx); \tag{237}$$
$$d=\text{E}^{\frac{0}{2}}-\text{E}^{-\frac{0}{2}}, \tag{238}$$
$$d\phi x=\phi\left(x+\frac{1}{2}\,dx\right)-\phi\left(x-\frac{1}{2}\,dx\right). \tag{239}$$

Of these, (235) and (237) are known forms.

71. Of the three chief vanishing fractions, with the expressions corresponding to them just given, the upper fraction will no doubt in most cases be found the most useful in practice, as being, on the whole, the simplest. Nevertheless, the central fraction (233) and the corresponding differential expression (239) will be found well worthy of attention on account of their symmetrical form. It cannot be doubted that cases will arise in which this quality of symmetry will prove an important aid to the analyst. To illustrate another advantage possessed by the central formulæ, let it be required to find $d\,(x^3)$. By the usual method,

$$d\,(x^3)=3x^2dx+3x\,(dx)^2+(dx)^3, \tag{240}$$

and by the central method,

$$d\,(x^3)=3x^2dx+\frac{1}{4}\,(dx)^3. \tag{241}$$

Here there is obviously less to be disregarded, and so far there is an advantage, even though it be only in appearance. Apart from all practical advantages, however, the consideration of the central formulæ cannot but be useful in affording a broader view of the subject than that usually taken. The same remark applies, of course, with still greater force to the general formulæ of paragraphs 68 and 69, not to speak of others still to be presented.

72. In the special cases thus far examined of the general differentiate-expression (220), we have supposed $h=0$, with a finite. Let us now consider the case in which $h=0$ and a is infinite. Let $a=-\frac{c}{h}$, so that $\partial=\text{E}^c\,\frac{\text{E}^h-1}{h}{}_{[h=0]}=\text{DE}^c$, where c has any finite value other than 0. Then $ah\partial=-c\text{DE}^c$, and we have the following series,

$$\text{D} = \text{DE}^c - c\text{D}^2\text{E}^{2c} + \frac{3}{2}c^2\text{D}^3\text{E}^{3c} - \frac{4^2}{2\,.\,3}c^3\text{D}^4\text{E}^{4c} + \frac{5^3}{2\,.\,3\,.\,4}c^4\text{D}^5\text{E}^{5c} - \ldots, \tag{242}$$

$$\text{D}\phi x = \text{D}\phi\,(x+c) - c\text{D}^2\phi\,(x+2c) + \frac{3}{2}c^2\text{D}^3\phi\,(x+3c) - \ldots. \tag{243}$$

As special cases,

$$\text{D}\phi x = \text{D}\phi\,(x+1) - \text{D}^2\phi\,(x+2) + \frac{3}{2}\text{D}^3\phi\,(x+3) - \ldots, \tag{244}$$

$$\text{D}\phi x = \text{D}\phi\,(x-1) + \text{D}^2\phi\,(x-2) + \frac{3}{2}\text{D}^3\phi\,(x-3) + \ldots. \tag{245}$$

If we divide both members of (242) by DE^c, and, putting $h = -c$, operate on ϕx, also on $\phi 0$, we shall have

$$\phi\,(x+h) = \phi x + h\text{D}\phi\,(x-h) + \frac{3}{2}h^2\text{D}^2\phi\,(x-2h) + \ldots, \tag{246}$$

$$\phi h = \phi 0 + h\phi'(-h) + \ldots, \tag{247}$$

where $\phi' x = \text{D}\phi x$. Though interesting, and probably new, these various series are comparatively unimportant.

73. Much more worthy of attention are those series, expressing the differentiate in terms of finite differences, which are derived from the general differentiate-expression (220) by giving to h some value other than 0. The principal value which h may assume is 1, and the formulæ derived for that value can be made to yield, by a suitable alteration of the variable, all the results obtainable by assigning to h any other value. When $h = 1$, we have the following general theorem for expressing a differentiate in terms of differences:

$$\text{D} = \partial + \frac{2a-1}{2}\partial^2 + \frac{3a-1}{2}\,\frac{3a-2}{3}\partial^3 + \ldots. \tag{248}$$

Here $\partial = \text{E}^{1-a} - \text{E}^{-a}$, and $\partial\phi x = \phi\,(x-a+1) - \phi\,(x-a)$. For example,

$$\partial c^x = c^x(c^{1-a} - c^{-a}) = c^x z\text{, } suppose, \tag{249}$$

$$\partial^2 c^x = c^x z^2, \tag{250}$$

$$\text{D}c^x = c^x\left(z + \frac{2a-1}{2}z^2 + \frac{3a-1}{2}\,\frac{3a-2}{3}z^3 + \ldots\right) = c^x\log c, \tag{251}$$

by (43).

74. In this case again, as with the general vanishing fraction (221), we find three principal values for a, namely, $a = 0$, $a = 1$, and $a = \frac{1}{2}$. Substituting these values respectively, we obtain three series, all more or less well known, expressing a differentiate in terms of what we may call upper, lower, and central differences. These are,

$$\text{D} = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots, \tag{252}$$

$$\text{D} = {}'\Delta + \frac{1}{2}\,{}'\Delta^2 + \frac{1}{3}\,{}'\Delta^3 + \ldots, \tag{253}$$

$$\text{D} = \Lambda - \frac{\Lambda^3}{8.3} + \frac{3}{2}\,\frac{\Lambda^5}{8^2.5} - \frac{3.5}{2.3}\,\frac{\Lambda^7}{8^3.7} + \frac{3.5.7}{2.3.4}\,\frac{\Lambda^9}{8^4.9} - \ldots. \tag{254}$$

There is also a known series expressing D in terms of mean central differences, which may be derived as follows from (254). Let $\text{I} = \frac{1}{2}(\text{E}^{\frac{1}{2}} + \text{E}^{-\frac{1}{2}})$; then $\text{I} = (1 + \frac{1}{4}\Lambda^2)^{\frac{1}{2}}$, and, by expansion,

$$\text{DI}^{-1} = \left(\Lambda - \frac{\Lambda^3}{8.3} + \ldots\right)\left(1 - \frac{\Lambda^2}{8} + \ldots\right) = \Lambda - \frac{\Lambda^3}{6} + \frac{\Lambda^5}{30} - \ldots, \tag{255}$$

and

$$\text{D} = \text{I}\Lambda - \frac{\text{I}\Lambda^3}{6} + \frac{\text{I}\Lambda^5}{30} - \frac{\text{I}\Lambda^7}{140} + \ldots. \tag{256}$$

75. The principal use to which these series have hitherto been put is to determine the value of a differentiate from given values of the function differentiated. The simplest possible illustration is probably as follows. Let us first construct a table of the values of x^2, and of their differences, from $x = -1$ to $x = 3$.

x	x^2	∂x^2	$\partial^2 x^2$	$\partial^3 x^2$
3	9		2	
		5		0
2	4	[4]	2	[0]
		3		0
1	1		2	
		1		0
0	0		2	
		-1		0
-1	1		2	

We see that $\Delta(-1)^2 = -1$, $\Delta^2(-1)^2 = 2$, $\Delta^3(-1)^2 = 0$; that ${}'\Delta 3^2 = 5$, ${}'\Delta^2 3^2 = 2$, ${}'\Delta^3 3^2 = 0$; that $\Lambda\left(\frac{3}{2}\right)^2 = 3$, $\Lambda^3\left(\frac{3}{2}\right)^2 = 0$; and that $\text{I}\Lambda 2^2 = 4$, $\text{I}\Lambda^3 2^2 = 0$. Then applying the four series in question, respectively, we find

$$\text{D}(-1)^2 = -1 - \frac{2}{2} = -2, \tag{257}$$

$$\text{D}3^2 = 5 + \frac{2}{2} = 6, \tag{258}$$

$$\text{D}\left(\frac{3}{2}\right)^2 = 3, \tag{259}$$

$$\text{D}2^2 = 4. \tag{260}$$

This use of differences is especially important when, for any reason, it is desired to find the differentiate of a function of which certain arithmetical values are ascertained, but of which the law is unknown.

76. Besides such customary uses, these series, and particularly those expressed in terms of Δ and $'\Delta$, will, I think, be found of great value towards the elementary explanation of $\mathrm{D} = \log \mathrm{E}$. No student, informed that a differentiate is a series of differences, can fail to understand the statement. It is requisite to introduce the idea of infinity in some form, and these series will be found at least as intelligible as the vanishing fractions, and worthy of a place beside them in explanatory statements; essentially necessary, indeed, to complete a comprehensive view of the subject. It is certainly as easy to understand $\mathrm{D}x^2 = \Delta x^2 - \frac{1}{2}\Delta^2 x^2 = 2x$ as $\mathrm{D}x^2 = \frac{(x+h)^2 - x^2}{h}_{[h=0]} = 2x$. I would call attention to the fact that (253) supplies a verbal definition of a differentiate which may readily be borne in mind, namely, *the sum of divided lower differences.**

77. The application of these series to given forms of function will afford useful exercise to the student. To a^x, for example, any of them may at once be applied in the manner already exhibited. As another example, let $\phi x = x^m$. By the binomial theorem,

$$\mathrm{D}_0 (x+0)^m = \mathrm{D}_0 x^m + \mathrm{D}_0 0 m x^{m-1} + \mathrm{D}_0 0^2 m \frac{m-1}{2} x^{m-2} + \ldots . \tag{261}$$

Now $\Delta_0 (x+0)^m = (x+1)^m - x^m = \Delta x^m$, and similarly for second and higher differences; hence

$$\mathrm{D}_0 (x+0)^m = \left(\Delta_0 - \frac{1}{2}\Delta_0^2 + \ldots\right)(x+0)^m = \left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right) x^m = \mathrm{D}x^m. \tag{262}$$

Also, $\mathrm{D}_0 x^m = (\Delta_0 - \ldots)\, x^m = 0$, and $\mathrm{D}_0 0 = (\Delta_0 - \ldots)\, 0 = 1$. As regards $\mathrm{D}_0 0^2$, $\mathrm{D}_0 0^3$, &c., we have it proved algebraically (De Morgan, *Calculus*, p. 255) that $\frac{\Delta^k 0^r}{k} = \Delta^{k-1} 0^{r-1} + \Delta^k 0^{r-1}$; and hence, when $r > 1$, we find by successive substitution that

$$\mathrm{D}_0 0^r = \left(\Delta_0 - \frac{1}{2}\Delta_0^2 + \ldots\right) 0^r = 0. \tag{263}$$

* De Morgan gives, in the article *Differential Calculus* of the *Penny Cyclopædia*, certain comparative illustrations of the current definitions of $\mathrm{D}x^3$, according to the systems of Infinitesimals, Prime and Ultimate Ratios, Fluxions, and Limits, and the Residual Analysis of Landen. Each of these definitions requires several lines of print, the nearest approach to an equation being

$$\mathrm{D}x^3 = \frac{x^3 - x^3}{x - x} = 3x^2.$$

Substituting these several results in (261), we have, finally,

$$\mathrm{D}x^m = mx^{m-1}. \tag{264}$$

78. We have now passed in review the more important equivalents for $\mathrm{D} = \log \mathrm{E}$ which may be deduced from the general differentiate-expression (220). We have seen how that theorem includes not only the known series in terms of Δ, $'\Delta$, and Λ, but also series in terms of an infinite number of other kinds of differences, besides the series in terms of DE^c; and how, in addition to such series, it comprehends an unlimited number of symbolic vanishing fractions, equivalents of D, one of which fractions exhibits, when applied in practice, the hitherto customary process of differentiation. Wide as that expression is, we shall shortly see that it is but a special case of a more comprehensive formula for the transformation of D^n; and, still further, we shall find that this latter formula is itself merely one case of a broader proposition regarding ∂^n, which again is but a special case of a general theorem relating to all functions of E. For the due presentation of this general theorem I find it necessary to lay down a new theory of factorials.

VI. *Theory of Factorials.*

79. Let

$$x^{[m]} = h^{m-1}x\,\frac{\Gamma(xh^{-1} + am)}{\Gamma(xh^{-1} + am - m + 1)}, \tag{265}$$

where a and h have the same value as in ∂, the difference-ratio symbol.* Whenever it is necessary to consider at the same time expressions involving more than one value of a or h, I shall add accents. Thus, while ∂ and $x^{[m]}$ are functions of a and h, ∂' and $x^{[m]'}$ will be the same functions of a' and h', and ∂'' and $x^{[m]''}$ those of a'' and h'', where a' and h', a'' and h'', may or may not be the same as a and h. Let us call (265) the general form of Primary Factorials; $x^{[3]}$, for example, being the general form of the primary factorial of the third degree.

80. By performing the operation, we find that

$$\partial x^{[m]} = mx^{[m-1]}. \tag{266}$$

81. Let us, except when otherwise expressly stated, consider only those cases in which m is neither negative nor fractional. We find that $x^{[0]} = 1$, that $x^{[1]} = x$, and that for all values of m greater than 1,

$$x^{[m]} = x\,(x + amh - h)(x + amh - 2h)\ldots(x + [a-1]\,mh + h)\,, \tag{267}$$

* Here also (see note to paragraph 65) more symmetrical expressions can be had by writing $\frac{1}{2}(1-b)$ for a.

where a and h may have any value, positive or negative, whole or fractional. For example, let $a = 17$ and $h = -\frac{1}{3}$; then

$$x^{[4]} = x\left(x - 22\tfrac{1}{3}\right)(x - 22)\left(x - 21\tfrac{2}{3}\right). \tag{268}$$

When $a = 0$, we have as a special case those functions to which, for positive indices, the name factorial has heretofore been confined.

82. When $h = 0$, and a is finite, we have from (267)

$$x^{[m]} = x^m. \tag{269}$$

When $h = 0$, and $a = -\frac{c}{h}$, where c is a finite quantity other than 0, we derive a remarkable function,

$$x^{[m]} = x\,(x - cm)^{m-1}. \tag{270}$$

When h is any quantity except 0 and 1, the expression $x^{[m]}$ may be reduced, by a suitable alteration of the variable, to a factorial form in which $h = 1$. I shall, therefore, for brevity, omit the consideration of such values of h. The chief special cases when $h = 1$ are those where $a = 0$, $a = 1$, and $a = \frac{1}{2}$, respectively,

$$x^{[m]} = x\,(x-1)(x-2)\ldots(x-m+1), \tag{271}$$

$$x^{[m]} = x\,(x+m-1)(x+m-2)\ldots(x+1), \tag{272}$$

$$x^{[m]} = x\left(x + \tfrac{1}{2}m - 1\right)\left(x + \tfrac{1}{2}m - 2\right)\ldots\left(x - \tfrac{1}{2}m + 1\right). \tag{273}$$

The last form is no doubt novel. We may call these three varieties of factorials upper, lower, and central, respectively, to correspond with the analogous difference operations, Δ, $'\Delta$, and Λ; and I would suggest for them the special symbols $x^{m)}$, $x^{(m}$, and $x^{(m)}$, respectively. It is scarcely necessary to say that whatever may be proved true in general of ∂ and $x^{[m]}$ will hold good of D and x^m, Δ and $x^{m)}$, $'\Delta$ and $x^{(m}$, Λ and $x^{(m)}$, DE^c and $x\,(x - cm)^{m-1}$, as well as all other possible special cases.

83. By repetition of (266),

$$\partial^n x^{[m]} = m^{n)} x^{[m-n]} = \frac{\Gamma(1+m)}{\Gamma(1+m-n)}\, x^{[m-n]}. \tag{274}$$

Operating on both sides by ∂^{-n}, dividing by $m^{n)}$, and writing $m + n$ for m, we have

$$\partial^{-n} x^m = \frac{\Gamma(1+m)}{\Gamma(1+m+n)}\, x^{[m+n]}, \tag{275}$$

showing that (274) is true, as a principal form, when n is negative. If $x = 0$, and $n < m$,

$$\partial^n 0^{[m]} = 0; \tag{276}$$

and when $n = m$,

$$\partial^m 0^{[m]} = \Gamma(1+m). \tag{277}$$

Operating on this with ∂^{n-m}, supposing $n > m$, we have

$$\partial^n 0^{[m]} = 0, \tag{278}$$

because ∂ (*constant*) $= 0$. When, therefore, n is either greater or less than m, both being integral,

$$\partial^n 0^{[m]} = 0. \tag{279}$$

If ϕx can be expressed in positive integral powers, one of the terms being $a_m x^m$,

$$\phi\partial 0^{[m]} = a_m \partial^m 0^{[m]} = a_m \Gamma(1+m) = a_m \partial'^m 0^{[m]'} = \phi\partial' 0^{[m]'}. \tag{280}$$

Again,

$$\phi(c\partial)\, 0^{[m]} = c^m a_m \partial^m 0^{[m]} = c^m \phi\partial 0^{[m]} = c^m \phi\partial' 0^{[m]'}. \tag{281}$$

84. Any factorial $x^{[m]}$ may be expressed in factorials of any other form, such as $x^{[m]'}$; for $x^{[m]}$ is by definition, let us say, $x^m + c_m x^{m-1} + \ldots$, and $x^{[m]'}$ is similarly $x^m + c_m' x^{m-1} + \ldots$, wherefore $x^{[m]}$, an algebraic expression of the mth degree, may be replaced by $x^{[m]'}$, also of the mth degree, plus factorials of lower degrees, the coefficients of which may be determined from the data, which are sufficient for that purpose. In general, therefore, when $n > m$,

$$\partial'^n 0^{[m]} = \partial'^n (0^{[m]'} + \ldots) = 0, \tag{282}$$

and when $n = m$,

$$\partial'^m 0^{[m]} = \partial'^m 0^{[m]'} = \Gamma(1+m). \tag{283}$$

85. Again,

$$\phi\partial 0^{[m]} = a_m \partial^m 0^{[m]} = a_m \partial^{m-1} \partial 0^{[m]} = a_m m \partial^{m-1} 0^{[m-1]} = \phi'\partial 0^{[m-1]}, \tag{284}$$

and by repetition,

$$\phi\partial 0^{[m]} = \phi^n \partial 0^{[m-n]} = \phi^n \partial' 0^{[m-n]'}, \tag{285}$$

where $\phi^n x = \mathrm{D}^n \phi x$. For example,

$$\varepsilon^{h\partial} 0^{[m]} = h^m \varepsilon^{h\partial} 0^{[0]} = h^m (1 + h\partial + \ldots)\, 0^{[0]} = h^m. \tag{286}$$

It follows from (285) and (152) that, when $\phi\partial$ can be expressed in positive integral powers of ∂,

$$\phi\partial 0^{[m]} = \phi^m \mathrm{D} 0^0 = \phi^m 0 = \mathrm{D}^m \phi 0. \tag{287}$$

86. Suppose

$$\phi\mathrm{E} = \psi\partial = a_0 + a_1\partial + a_2\partial^2 + \ldots. \tag{288}$$

By Herschel's theorem (162),

$$\psi\partial = \psi\mathrm{D}0^0 + \psi\mathrm{D}0\,.\,\partial + \frac{1}{2}\,\psi\mathrm{D}0^2\,.\,\partial^2 + \ldots, \tag{289}$$

and by Maclaurin's theorem (120),

$$\psi\partial = \mathrm{D}^0\psi 0 + \mathrm{D}\psi 0\,.\,\partial + \frac{1}{2}\,\mathrm{D}^2\psi 0\,.\,\partial^2 + \ldots. \tag{290}$$

From (288) we have at once, since $\phi\mathrm{E}0^{[m]} = a_m \partial^m 0^{[m]} = 2.3.4\ldots m a_m$, and therefore $a_m = \frac{1}{2.3.4\ldots m}\,\phi\mathrm{E}0^{[m]}$,

$$\phi\text{E} = \phi\text{E}0^{[0]} + \phi\text{E}0^{[1]}\,.\,\partial + \frac{1}{2}\,\phi\text{E}0^{[2]}\,.\,\partial^2 + \frac{1}{2\,.\,3}\,\phi\text{E}0^{[3]}\,.\,\partial^3 + \dots \tag{291}$$

The same result may be had from (289) or (290), observing respectively (280) or (287). This is the general theorem referred to in paragraph 78. It may fitly be named *the factorial theorem.* It includes as special cases a very large number of known propositions of the Differential Calculus and Calculus of Finite Differences, and affords a ready instrument for the discovery of relations hitherto unnoticed. It should be laid down among the earliest propositions in any formal treatise on the Calculus. Recurring to the line of thought pursued in paragraph 52, we may remark that this theorem applies in all cases where ϕE produces determinate results, since it holds good, as proved, for all functions of E which can be expressed in positive integral powers of ∂. The following is a variation which results from (281):

$$\psi\,(c\partial) = \psi\partial' 0^{[0]'} + \psi\partial' 0^{[1]'}\,.\,c\partial + \frac{1}{2}\,\psi\partial' 0^{[2]'}\,.\,c^2\partial^2 + \dots \tag{292}$$

Here ∂' and $x^{[m]'}$ may or may not be the same as ∂ and $x^{[m]}$, and c may have any value. Of course, $\phi\text{E}0^{[0]} = \phi 1$, and $\phi\partial 0^{[0]} = \phi 0$.

87. An important case of the factorial theorem is that where $\phi\text{E} = \text{E}^k$. Applying it to ψx, we have the following *generalization of Taylor's theorem:*

$$\psi\,(x + k) = \psi x + k\partial\psi x + \frac{1}{2}\,k^{[2]}\partial^2\psi x + \dots \tag{293}$$

If $\psi x = x^{[m]'}$, we obtain a *generalization of the binomial theorem*, true for all values of m, including negative and fractional values,

$$(x + k)^{[m]'} = x^{[m]'} + k\partial x^{[m]'} + \frac{1}{2}\,k^{[2]}\partial^2 x^{[m]'} + \dots \tag{294}$$

This enables us to expand any binomial factorial in factorials of any other desired form. For example, to expand $(x+k)^{[m]}$ in factorials of k, we have this minor generalization of the binomial theorem,

$$(x + k)^{[m]} = x^{[m]} + mx^{[m-1]}k + m\,\frac{m-1}{2}\,x^{[m-2]}k^{[2]} + \dots, \tag{295}$$

good for all values of m; or, to expand $(x + k)^m$ in general primary factorials,

$$(x + k)^m = x^m + k\partial x^m + \frac{1}{2}\,k^{[2]}\partial^2 x^m + \dots \tag{296}$$

If $x = 0$, we have from (294), for the expansion of a factorial in factorials of any other kind, whether m be positive or negative, whole or fractional,

$$k^{[m]'} = k\partial 0^{[m]'} + \frac{1}{2}\,k^{[2]}\partial^2 0^{[m]'} + \dots \tag{297}$$

If, in (293), $x = 0$, we have a *generalization of Maclaurin's theorem,*

$$\psi k = \psi 0 + k\partial\psi 0 + \frac{1}{2}\,k^{[2]}\partial^2\psi 0 + \dots \tag{298}$$

These various theorems for factorial expansion will be found capable of many useful and interesting applications. For example, in (298) let $\psi k = c^k$, and let us write x for k; then

$$c^x = 1 + x\partial_0 c^0 + \frac{1}{2}x^{[2]}\partial_0^2 c^0 + \ldots, \tag{299}$$

a *generalization of the exponential theorem.* An expression for the value of log $(1+x)$ may similarly be derived, of which one case is this known series,

$$\log(1+x) = \log 2 . x + \frac{1}{2}\log\frac{3}{2^2} . x(x-1) + \ldots. \tag{300}$$

If, in (299), we write ε for c and $k\text{D}$ for x, we have this result,

$$\text{E}^k = 1 + \partial_0\varepsilon^0 . k\text{D} + \ldots, \tag{301}$$

$$\phi(x+k) = \phi x + \partial_0\varepsilon^0 . k\text{D}\phi x + \frac{1}{2}\partial_0^2\varepsilon^0 . (k\text{D})^{[2]}\phi x + \ldots. \tag{302}$$

88. A certain variation of (299) is so remarkable as to be worthy of extended notice. Let g be such that $\partial g^x = g^x$, that is to say, that

$$\frac{g^{x+(1-a)h} - g^{x-ah}}{h} = g^x; \tag{303}$$

then

$$\frac{g^{(1-a)h} - g^{-ah}}{h} = 1, \tag{304}$$

whence

$$g^h - 1 = hg^{ah}. \tag{305}$$

The solution of one of these equations will give the value of g. If more than one solution presents itself, that only can be accepted which agrees with the condition $\partial g^x = g^x$. When $h = 0$, equation (304) becomes $\log g = 1$, or $g = \varepsilon$. From (299) we have at once the series in question,

$$g^x = 1 + x + \frac{1}{2}x^{[2]} + \frac{1}{2.3}x^{[3]} + \ldots, \tag{306}$$

of which the exponential theorem (23) is a special case. If $h = 1$ and $a = 0$, we have a series verifiable by expanding $(1+1)^x$,

$$2^x = 1 + x + \frac{1}{2}x^{2)} + \ldots. \tag{307}$$

If $h = 1$ and $a = \frac{1}{2}$,

$$\left(\frac{3+\sqrt{5}}{2}\right)^x = 1 + x + \frac{1}{2}x^{(2)} + \ldots. \tag{308}$$

If $h = 2$ and $a = \frac{1}{2}$,

$$(1+\sqrt{2})^x = 1 + x + \frac{1}{2}x^{[2]} + \ldots, \tag{309}$$

where $x^{[2]}=x^2$, $x^{[3]}=x(x+1)(x-1)$, $x^{[4]}=x(x+2)x(x-2)$, $x^{[5]}=x(x+3)(x+1)(x-1)(x-3)$, and so on. If $h=\frac{1}{2}$ and $a=0$,

$$\left(\frac{9}{4}\right)^x=1+x+\frac{1}{2}x^{[2]}+\ldots, \tag{310}$$

where $x^{[2]}=x\left(x+\frac{1}{2}\right)$, $x^{[3]}=x\left(x-\frac{1}{2}\right)(x-1)$, and so on. If $h=\frac{1}{2}$ and $a=1$, similarly,

$$4^x=1+x+\frac{1}{2}x^{[2]}+\ldots, \tag{311}$$

where $x^{[2]}=x\left(x+\frac{1}{2}\right)$, $x^{[3]}=x\left(x+\frac{1}{2}\right)(x+1)$, and so on. It is needless to multiply these illustrations, which show that ε is but a type of an infinite number of constants, and that the exponential theorem is but the corresponding type of an infinite number of factorial series of the same general form. The series above given may be verified, of course, by arithmetical approximation. It may be shown further that ε^x itself is capable of an unlimited number of factorial expansions, all having the same coefficients as the exponential theorem. In general,

$$\varepsilon^x=1+x+\frac{1}{2}x^{[2]}+\frac{1}{2\,.\,3}x^{[3]}+\ldots, \tag{312}$$

where $a=h^{-1}\log\frac{\varepsilon^h-1}{h}$. Thus, if $h=1$, $a=\log(\varepsilon-1)$, and $x^{[2]}=x(x+2a-1)$, $x^{[3]}=x(x+3a-1)(x+3a-2)$, and so on; if $h=\log(1+h)=0.01$ nearly, $a=0$, and $x^{[2]}=x(x-h)$, $x^{[3]}=x(x-h)(x-2h)$, and so on; and if $h=\log(1+h\varepsilon)=1.75$ nearly, $a=h^{-1}$, and $x^{[2]}=x(x+2-h)$, $x^{[3]}=x(x+3-h)(x+3-2h)$, and so on.

89. Let z be that function of x which ∂ is of E, namely, $\frac{x^{(1-a)h}-x^{-ah}}{h}$, so that $x=(1+hzx^{ah})^{\frac{1}{h}}$. Since $\phi\text{E}_0x^0=\phi x$, $\partial_0^n x^0=z^n$. Applying the factorial theorem to x^0, we obtain a *generalization of Herschel's theorem*,

$$\phi x=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{[2]}+\ldots. \tag{313}$$

If $\phi x=\psi z$, $\phi\text{E}=\psi\partial$. If $h=0$, $z=\log x$, $x=\varepsilon^z$, $0^{[n]}=0^n$, and we have, as a special case, Herschel's theorem. If $h=1$ and $a=0$, $x=1+z$, and

$$(1+z)=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{2)}+\ldots; \tag{314}$$

while if $h=1$ and $a=1$, $x=\frac{1}{1-z}$, and

$$\phi\frac{1}{1-z}=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{(2}+\ldots. \tag{315}$$

To illustrate these apparently novel expansion-theorems, take

$$(1+z)^n = \mathrm{E}^n 0^0 + z\mathrm{E}^n 0 + \frac{1}{2} z^2 \mathrm{E}^n 0^{2)} + \dots$$

$$= 1 + zn + \frac{1}{2} z^2 n^{2)} + \dots, \tag{316}$$

$$\left(\frac{1}{1-z}\right)^n = 1 + zn + \frac{1}{2} z^2 n^{(2} + \dots, \tag{317}$$

as by the binomial theorem. By putting $h = zx$, we easily derive from (314) and (315) two *variations of Taylor's theorem:*

$$\phi(x+h) = \phi(x\mathrm{E})0^0 + hx^{-1}\phi(x\mathrm{E})0 + \frac{1}{2} h^2 x^{-2} \phi(x\mathrm{E}) 0^{2)} + \dots, \tag{318}$$

$$\phi(x+h) = \phi\frac{x}{\mathrm{E}} 0^0 - hx^{-1}\phi\frac{x}{\mathrm{E}} 0 + \frac{1}{2} h^2 x^{-2} \phi\frac{x}{\mathrm{E}} 0^{(2} - \dots. \tag{319}$$

These formulæ are particularly worthy of attention. In them, we have Taylor's theorem demonstrated, and the coefficients expressed, without any reference whatever to the operation of differentiation; a result, the possibility of which would have seemed incredible. We obtain by this means expressions equivalent to $\mathrm{D}\phi x$ which may be added, with advantage, to the list of those already considered; the chief advantage being that they are neither vanishing fractions nor infinite series, and cannot, indeed, be looked upon as in any respect transcendental. They are,

$$\mathrm{D}\phi x = x^{-1}(x\mathrm{E})\,0, \tag{320}$$

$$\mathrm{D}\phi x = -x^{-1}\phi\frac{x}{\mathrm{E}}\,0. \tag{321}$$

For example,

$$\mathrm{D}\epsilon^x = x^{-1}\epsilon^{x+x\Delta}0 = x^{-1}\epsilon^x(1 + x\Delta + \dots)\,0 = \epsilon^x, \tag{322}$$

$$\mathrm{D}\log x = x^{-1}(\log x + \log[1+\Delta])\,0 = x^{-1}\left(\Delta - \frac{1}{2}\Delta^2 + \dots\right)0 = x^{-1}, \tag{323}$$

$$\mathrm{D}x^m = x^{-1}x^m\mathrm{E}^m 0 = mx^{m-1}. \tag{324}$$

These expressions for $\mathrm{D}\phi x$ may be proved independently of Taylor's theorem. Thus, if $a_m x^m$ be the general term of ϕx, that of $\mathrm{D}\phi x$ will be $a_m m x^{m-1}$, which is also that of $x^{-1}\phi(x\mathrm{E})\,0$, or $x^{-1}a_m x^m \mathrm{E}^m 0$. In general, similarly,

$$\mathrm{D}^n\phi x = x^{-n}\phi(x\mathrm{E})\,0^{n)}, \tag{325}$$

$$\mathrm{D}^n\phi x = (-1)^n x^{-n}\phi\frac{x}{\mathrm{E}}\,0^{(n}. \tag{326}$$

For example, n being a positive integer,

$$\mathrm{D}^n \log x = x^{-n}(\log x + \log[1+\Delta])\,0^{n)} = x^{-n}(-1)^{n-1}\frac{1}{n}\Delta^n 0^{n)}.* \tag{237}$$

* Since this was written, I have found other forms of $\mathrm{D}^n\phi x$, and therefore of Taylor's theorem. By (110), $\phi\mathrm{E}_0 c^0 = \phi c$; hence, $\phi(x+\Delta_0)(1+h)^0 = \phi(x+h)$, and

$$\phi(x+h) = \phi x + h\phi(x+\Delta)\,0 + \frac{1}{2}h^2\phi(x+\Delta)\,0^{2)} + \dots.$$

90. If, in (313), we write ϕ log for ϕ, we have a logarithmic formula of extraordinary generality, which may properly be called *the logarithmic theorem*:

$$\phi \log x = \phi \mathrm{D}0^0 + z\phi \mathrm{D}0 + \frac{1}{2} z^2 \phi \mathrm{D}0^{[2]} + \ldots \tag{328}$$

An obviously important case is that where $\phi \log x = (\log x)^n$, and of this, again, the most common and most useful special case is

$$\log x = z + \frac{1}{2} z^2 \mathrm{D}0^{[2]} + \frac{1}{2\,.\,3} z^3 \mathrm{D}0^{[3]} + \ldots \tag{329}$$

We have thus, though in a different and more perspicuous shape, the general logarithmic series (31). To reduce it to the algebraic form of (31), we have only to put $y = hz$, and to observe that

$$\mathrm{D}0^{[2]} = \mathrm{D}_0 0\,(0 + 2ah - h) = 2ah - h, \tag{330}$$

and so on, the general rule being

$$\mathrm{D}0^{[n]} = \mathrm{D}0\phi 0 = 0\mathrm{D}\phi 0 + \phi 0 \mathrm{D}0 = \phi 0. \tag{331}$$

91. Of the various functions of E which may be expressed by the aid of the factorial theorem in terms of ∂, the most important are those other difference-ratios, of whatever degree, collectively represented by the symbol ∂'^n. When $\phi \mathrm{E} = \partial'^n$, all terms of the expansion prior to that containing $\partial'^n 0^{[n]}$ vanish, as may be seen from (282), and we have remaining the following *difference-ratio transformation formula:*

$$\partial'^n = \partial^n + \frac{\partial'^n 0^{[n+1]}}{\Gamma(n+2)} \partial^{n+1} + \frac{\partial'^n 0^{[n+2]}}{\Gamma(n+3)} \partial^{n+2} + \ldots \tag{332}$$

Of this formula one or two special cases are already known, as where ∂ and ∂' are respectively D and Δ, and *vice versa;* though it does not appear to be admitted, in the discussion of those cases, that any negative value can be assigned to n.* I shall now show that the general formula, including, of course, the cases just referred to, holds good when n is negative. That $\partial\phi^{[0]} = 0\phi^{[-1]} = \frac{\Gamma 1}{\Gamma 0} x^{[-1]}$ follows from (274). If it be doubted, we may, for present purposes, define $\Gamma 0$ to be $\Gamma 1 \frac{x^{[-1]}}{\partial x^{[0]}}$, $\Gamma(-1)$ to be $\frac{\Gamma 0}{-1}$, and so on, so that (274) may hold good for all possible integral values of m and n. The quantities so defined are imaginary, indeed, but if their use enables us to reach

Here $\mathrm{D}^n\phi x = \phi(x + \Delta)\, 0^{n)}$. Similarly, $\phi(x + {}^{\backprime}\Delta_0)(1-h)^{-0} = \phi(x + h)$, and $\mathrm{D}^n\phi x = \phi(x + {}^{\backprime}\Delta)\, 0^{(n}$. In general, $\mathrm{D}^n\phi x = \phi(x + \partial_0)\, 0^{[n]}$, and, still more generally, $h^n \mathrm{D}^n \phi(kx) = k^n \phi(kx + h\partial_0)\, 0^{[n]}$; an expression readily derived from (167) and (280). If $n = 1$, we have $\mathrm{D}\phi x = \phi(x + \partial_0)\, 0$. For example, $\mathrm{D}\varepsilon^{x^2} = \varepsilon^{x^2} \varepsilon^{2x\partial + \partial}\, 0 = \varepsilon^{x^2}(1 + 2x\partial + \ldots)\, 0 = \varepsilon^{x^2} 2x$.

* Compare Boole, *Finite Differences*, 2d ed., p. 24.

finite results, no valid objection can be made to it. We find by the factorial theorem, n being supposed negative, that

$$\left(\frac{\partial}{\partial'}\right)^{-n} = \left(\frac{\partial}{\partial'}\right)^{-n} 0^0 + \left(\frac{\partial}{\partial'}\right)^{-n} 0 \,.\, \partial + \frac{1}{2}\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[2]} \,.\, \partial^2 + \ldots . \qquad (333)$$

From (274),

$$\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[m]} = \partial'^{n} 0^{[m+n]} \frac{\Gamma(1+m)}{\Gamma(1+m+n)}. \qquad (334)$$

Making the proper substitutions, and multiplying both sides of (333) by ∂^n, we have (332) true when n is negative. That the coefficients are in that case finite may be seen, for

$$\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[m]} = \partial^{-n} \partial'^{n} 0^{[m]}, \qquad (335)$$

and since $0^{[m]}$ may be expressed in factorials of the form $0^{[m]'}$, neither the operation ∂'^{n} nor the subsequent operation ∂^{-n} can destroy its finite character.

92. The coefficient of ∂^{n+r} in (332) is $\frac{\partial'^{n} 0^{[n+r]}}{\Gamma(n+r+1)}$. This class of coefficients is likely to become so important that it will be desirable to assign to it a special symbol, by way of abbreviation. Let the general symbol be ${}^{[r]}\partial'^{n}$, and let the same device be employed in all special cases, so that, for example,

$$\frac{\mathrm{D}^n 0^{(n+r)}}{\Gamma(n+r+1)} = {}^{(r)}\mathrm{D}^n, \qquad (336)$$

$$\frac{\Lambda^n 0^{n+r}}{\Gamma(n+r+1)} = {}^{r}\Lambda^n, \qquad (337)$$

and so on. We may, therefore, wherever we see an index prefixed to a difference-ratio symbol, understand that it indicates a constant coefficient.

93. Using these symbols, we may thus write the difference-ratio transformation formula (332):

$$\partial'^{n} = \partial^n + {}^{[1]}\partial'^{n} \,.\, \partial^{n+1} + {}^{[2]}\partial'^{n} \,.\, \partial^{n+2} + \ldots . \qquad (338)$$

If $\partial' = \mathrm{D}$,

$$\mathrm{D}^n = \partial^n + {}^{[1]}\mathrm{D}^n \,.\, \partial^{n+1} + {}^{[2]}\mathrm{D}^n \,.\, \partial^{n+2} + \ldots, \qquad (339)$$

where, if $n = 1$,

$$\mathrm{D} = \partial + {}^{[1]}\mathrm{D}^1 \,.\, \partial^2 + {}^{[2]}\mathrm{D}^1 \,.\, \partial^3 + \ldots, \qquad (340)$$

which is merely a concise way of writing the general differentiate-expression (220). The interpretation of ${}^{[m]}\mathrm{D}^1$ has been illustrated in paragraph 90, in discussing the corresponding logarithmic series.

94. Perhaps the next most important special case of the transformation formula (332, 338) is that where $n = -1$,

$$\partial'^{-1} = \partial^{-1} + {}^{[1]}\partial'^{-1} \,.\, \partial^0 + {}^{[2]}\partial'^{-1} \,.\, \partial + \ldots, \qquad (341)$$

a most comprehensive *summation formula*, which includes as many special cases as there are ways of combining the special forms of ∂, such as D, Δ, $'\Delta$, Λ, &c., including, as we shall shortly see, those mean central differences whose symbol is $\mathrm{I}\Delta^n$. Among these special cases, which, having been thus distinctly indicated, it is needless to write out at present, are several known formulæ; known, that is to say, in substance, though the correct form of their coefficients is probably a novelty. One of these is the celebrated series of Maclaurin and Euler, namely, in our notation,

$$\Delta^{-1} = \mathrm{D}^{-1} + {}^{1}\Delta^{-1}.\,\mathrm{D}^{0} + {}^{2}\Delta^{-1}.\,\mathrm{D} + \ldots, \tag{342}$$

the coefficients of which are factors of the formerly inexplicable Numbers of Bernoulli.* We may write (341) in this form,

$$\partial'^{-1} = (1 + {}^{[1]}\partial'^{-1}.\,\partial + {}^{[2]}\partial'^{-1}.\,\partial^{2} + \ldots)\,\partial^{-1}, \tag{343}$$

whence

$$\partial'^{-1}\partial = 1 + {}^{[1]}\partial'^{-1}.\,\partial + {}^{[2]}\partial'^{-1}.\,\partial^{2} + \ldots. \tag{344}$$

Various special cases of these two formulæ will be found useful in practice. When $\partial = \Delta$ and $\partial' = \mathrm{D}$, we derive from (344) the well-known formula of Laplace customarily employed for mechanical quadrature.

95. The coefficients required in the more important applications of the factorial theorem ought to be tabulated. This is particularly true of those coefficients, represented by ${}^{[r]}\partial^{n}$, which are needed in applying the difference-ratio transformation formula (338), a theorem of which many cases will become increasingly important in the future. As a specimen of what should be done in the tabulation of coefficients, I give now a table of ${}^{r}\Delta^{n}$ and ${}^{r)}\mathrm{D}^{n}$, computed with due care, which will be found useful in applying these two formulæ,

$$\Delta^{n} = \mathrm{D}^{n} + {}^{1}\Delta^{n}.\,\mathrm{D}^{n+1} + {}^{2}\Delta^{n}.\,\mathrm{D}^{n+2} + \ldots, \tag{345}$$

and

$$\mathrm{D}^{n} = \Delta^{n} + {}^{1)}\mathrm{D}^{n}.\,\Delta^{n+1} + {}^{2)}\mathrm{D}^{n}.\,\Delta^{n+2} + \ldots. \tag{346}$$

These formulæ are deduced from (338) by putting $\partial = \mathrm{D}$ and $\partial' = \Delta$, and *vice versa*. They are well known for positive values of n, though the coefficients do not seem to have been tabulated; and are true, as already shown, for negative values. The table contains, of course, the coefficients of x^{n+r} in $(\varepsilon^{x} - 1)^{n}$ and $[\log(1+x)]^{n}$. When $r < 0$, ${}^{r}\Delta^{n} = {}^{r)}\mathrm{D}^{n} = 0$. [See Note, pp. 150 and 151.]

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* These numbers are, ${}^{1}\Delta^{-1} = -\frac{1}{2}$, $2\,.\,{}^{2}\Delta^{-1} = 2\Delta^{-1}0 = \frac{1}{6}$, $2.3.4.\,{}^{4}\Delta^{-1} = 4\Delta^{-1}0^{3} = -\frac{1}{30}$. $2.3.4.5.6.\,{}^{6}\Delta^{-1} = 6\Delta^{-1}0^{5} = \frac{1}{42}$, and so on.

As a mere object of curiosity, this table is very remarkable. An unlimited number of general formulæ exhibiting relations between the tabular numbers may be devised. Some fifty which I have noticed, including a small number derived from known expressions by dividing by $\Gamma(n+r+1)$, are appended. Negative values may be assigned to c.

Equivalents of ${}^{r}\varDelta^{n}$:

$${}^{r)}\mathrm{D}^{-n-r-1} - {}^{r-1}\varDelta^{n+1},$$

$$\frac{n+r+1}{n+1}\cdot{}^{r}\varDelta^{n+1} - {}^{r-1}\varDelta^{n+1},$$

$$-{}^{1)}\mathrm{D}^{n}\cdot{}^{r-1}\varDelta^{n+1} - {}^{2)}\mathrm{D}^{n}\cdot{}^{r-2}\varDelta^{n+2} - \ldots,$$

$$-\frac{n}{n+1}\cdot{}^{1}\varDelta^{-n-1}\cdot{}^{r-1}\varDelta^{n+1} - \frac{n}{n+2}\cdot{}^{2}\varDelta^{-n-2}\cdot{}^{r-2}\varDelta^{n+2} - \ldots,$$

$$\frac{1}{2^{r}-1}\left(n\frac{{}^{r-1}\Delta^{n+1}}{2} + n\frac{n-1}{2}\frac{{}^{r-2}\Delta^{n+2}}{2^{2}} + n\frac{n-1}{2}\frac{n-2}{3}\frac{{}^{r-3}\Delta^{n+3}}{2^{3}} + \ldots\right),$$

$$(n+r+1)\left(\frac{{}^{r}\Delta^{n+1}}{n+1} - \frac{{}^{r-1}\Delta^{n+2}}{n+2} + \frac{{}^{r-2}\Delta^{n+3}}{n+3} - \ldots\right),$$

$${}^{r}\varDelta^{n+c} + {}^{1)}\mathrm{D}^{c}\cdot{}^{r-1}\varDelta^{n+c+1} + {}^{2)}\mathrm{D}^{c}\cdot{}^{r-2}\varDelta^{n+c+2} + \ldots,$$

$${}^{r}\varDelta^{n+c} + \frac{c}{c+1}\cdot{}^{1}\varDelta^{-c-1}\cdot{}^{r-1}\varDelta^{n+c+1} + \frac{c}{c+2}\cdot{}^{2}\varDelta^{-c-2}\cdot{}^{r-2}\varDelta^{n+c+2} + \ldots,$$

$$-{}^{1)}\mathrm{D}^{n+r-1}\cdot{}^{r-1}\varDelta^{n} - {}^{2)}\mathrm{D}^{n+r-2}\cdot{}^{r-2}\varDelta^{n} - \ldots,$$

$$-\frac{n+r-1}{n+r}\cdot{}^{1}\varDelta^{-n-r}\cdot{}^{r-1}\varDelta^{n} - \frac{n+r-2}{n+r}\cdot{}^{2}\varDelta^{-n-r}\cdot{}^{r-2}\varDelta^{n} - \ldots,$$

$$\frac{n}{n+r}\left({}^{r}\varDelta^{n-1} + {}^{r-1}\varDelta^{n-1} + \frac{{}^{r-2}\Delta^{n-1}}{1.2} + \ldots\right),$$

$${}^{r}\varDelta^{n+c} + {}^{1}\varDelta^{-c}\cdot{}^{r-1}\varDelta^{n+c} + {}^{2}\varDelta^{-c}\cdot{}^{r-2}\varDelta^{n+c} + \ldots,$$

$${}^{r}\varDelta^{n+c} + \frac{c}{c-1}\cdot{}^{1)}\mathrm{D}^{c-1}\cdot{}^{r-1}\varDelta^{n+c} + \frac{c}{c-2}\cdot{}^{2)}\mathrm{D}^{c-2}\cdot{}^{r-2}\varDelta^{n+c} + \ldots,$$

$$r\cdot{}^{r}\varDelta^{n-c} - r\frac{r-1}{2}\cdot{}^{r}\varDelta^{n-2c} + \ldots + \left(\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r-1}\cdot{}^{1)}\mathrm{D}^{n+r-1}\cdot{}^{r-1)}\mathrm{D}^{1-n-r} - \frac{n}{n+r-2}\cdot{}^{2)}\mathrm{D}^{n+r-2}\cdot{}^{r-2)}\mathrm{D}^{2-n-r} - \ldots,$$

$$-\frac{n}{n+r}\left({}^{1}\varDelta^{-n-r}\cdot{}^{r-1)}\mathrm{D}^{1-n-r} + {}^{2}\varDelta^{-n-r}\cdot{}^{r-2)}\mathrm{D}^{2-n-r} + \ldots\right),$$

$$n\frac{n-1}{n+r}\left(\frac{{}^{r)}\mathrm{D}^{1-n-r}}{n+r-1} + \frac{{}^{r-1)}\mathrm{D}^{2-n-r}}{n+r-2} + \frac{{}^{r-2)}\mathrm{D}^{3-n-r}}{(n+r-3)\,1.2} + \ldots\right),$$

$$\frac{c-n}{c-n-r}\cdot{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n}{c-n-r+1}\cdot{}^{1}\varDelta^{c}\cdot{}^{r-1)}\mathrm{D}^{c-n-r+1} + \ldots,$$

$$\frac{c-n}{c-n-r}\cdot{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n}{c-n-r+1}\,\frac{c}{c+1}\cdot{}^{1)}\mathrm{D}^{-c-1}\cdot{}^{r-1)}\mathrm{D}^{c-n-r+1} + \ldots,$$

$$r\,\frac{c-n}{c-n-r}\cdot{}^{r)}\mathrm{D}^{c-n-r} - r\frac{r-1}{2}\,\frac{2c-n}{2c-n-r}\cdot{}^{r)}\mathrm{D}^{2c-n-r} + \ldots + \left(\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r}\left({}^{1}\varDelta^{-n-1}\cdot{}^{r-1)}\mathrm{D}^{-n-r} + {}^{2}\varDelta^{-n-2}\cdot{}^{r-2)}\mathrm{D}^{-n-r} + \ldots\right),$$

$$-\frac{n+1}{n+r}\cdot{}^{1)}\mathrm{D}^{n}\cdot{}^{r-1)}\mathrm{D}^{-n-r} - \frac{n+2}{n+r}\cdot{}^{2)}\mathrm{D}^{n}\cdot{}^{r-2)}\mathrm{D}^{-n-r} - \ldots,$$

$$\frac{n}{n+r}\,\frac{1}{2^{r}-1}\left(\frac{n+1}{2}\cdot{}^{r-1)}\mathrm{D}^{-n-r} + \frac{n+2}{2^{2}}\,\frac{n-1}{2}\cdot{}^{r-2)}\mathrm{D}^{-n-r} + \frac{n+3}{2^{3}}\,\frac{n-1}{2}\,\frac{n-2}{3}\cdot{}^{r-3)}\mathrm{D}^{-n-r} + \ldots\right),$$

$${}^{r)}\mathrm{D}^{-n-r-1} - {}^{r-1)}\mathrm{D}^{-n-r-1} + {}^{r-2)}\mathrm{D}^{-n-r-1} - \ldots,$$

$$\frac{c-n}{c-n-r}\cdot{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n-1}{c-n-r}\cdot{}^{1)}\mathrm{D}^{-c}\cdot{}^{r-1)}\mathrm{D}^{c-n-r} + \ldots,$$

$$\frac{c-n}{c-n-r}\cdot{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n-1}{c-n-r}\,\frac{c}{c-1}\cdot{}^{1}\varDelta^{c-1}\cdot{}^{r-1)}\mathrm{D}^{c-n-r} + \ldots.$$

Equivalents of ${}^{r)}\mathrm{D}^{n}$:

$${}^{r}\varDelta^{-n-r-1} + {}^{r-1}\varDelta^{-n-r},$$

$$\frac{n+r+1}{n+1}\,.\,{}^{r)}\mathrm{D}^{n+1} + \frac{n+r}{n+1}\,.\,{}^{r-1)}\mathrm{D}^{n+1},$$

$$-{}^{1}\varDelta^{n}\,.\,{}^{r-1)}\mathrm{D}^{n+1} - {}^{2}\varDelta^{n}\,.\,{}^{r-2)}\mathrm{D}^{n+2} - \ldots,$$

$$-\frac{n}{n+1}\,.\,{}^{1)}\mathrm{D}^{-n-1}\,.\,{}^{r-1)}\mathrm{D}^{n+1} - \frac{n}{n+2}\,.\,{}^{2)}\mathrm{D}^{-n-2}\,.\,{}^{r-2)}\mathrm{D}^{n+2} - \ldots,$$

$$\frac{-1}{2^{r}-1}\left(\frac{n+r-1}{2}\,.\,{}^{r-1)}\mathrm{D}^{n} - \frac{n+r-2}{2^{2}}\,\frac{n+r+1}{2}\,.\,{}^{r-2)}\mathrm{D}^{n} + \frac{n+r-3}{2^{3}}\,\frac{n+r+1}{2}\,\frac{n+r+2}{3}\,.\,{}^{r-3)}\mathrm{D}^{n} - \ldots\right),$$

$$(n+r+1)\left(\frac{{}^{r)}\mathrm{D}^{n+1}}{n+1} + \frac{{}^{r-1)}\mathrm{D}^{n+2}}{n+2} + \frac{{}^{r-2)}\mathrm{D}^{n+3}}{(n+3)\,1.2} + \ldots\right),$$

$${}^{r)}\mathrm{D}^{n+c} + {}^{1}\varDelta^{c}\,.\,{}^{r-1)}\mathrm{D}^{n+c+1} + {}^{2}\varDelta^{c}\,.\,{}^{r-2)}\mathrm{D}^{n+c+2} + \ldots,$$

$${}^{r)}\mathrm{D}^{n+c} + \frac{c}{c+1}\,.\,{}^{1)}\mathrm{D}^{-c-1}\,.\,{}^{r-1)}\mathrm{D}^{n+c+1} + \frac{c}{c+2}\,.\,{}^{2)}\mathrm{D}^{-c-2}\,.\,{}^{r-2)}\mathrm{D}^{n+c+2} + \ldots,$$

$$-{}^{1}\varDelta^{n+r-1}\,.\,{}^{r-1)}\mathrm{D}^{n} - {}^{2}\varDelta^{n+r-2}\,.\,{}^{r-2)}\mathrm{D}^{n} - \ldots,$$

$$-\frac{n+r-1}{n+r}\,.\,{}^{1)}\mathrm{D}^{-n-r}\,.\,{}^{r-1)}\mathrm{D}^{n} - \frac{n+r-2}{n+r}\,.\,{}^{2)}\mathrm{D}^{-n-r}\,.\,{}^{r-2)}\mathrm{D}^{n} - \ldots,$$

$$\frac{n}{n+r}\left({}^{r)}\mathrm{D}^{n-1} - {}^{r-1)}\mathrm{D}^{n-1} + {}^{r-2)}\mathrm{D}^{n-1} - \ldots\right),$$

$${}^{r)}\mathrm{D}^{n+c} + {}^{1)}\mathrm{D}^{-c}\,.\,{}^{r-1)}\mathrm{D}^{n+c} + {}^{2)}\mathrm{D}^{-c}\,.\,{}^{r-2)}\mathrm{D}^{n+c} + \ldots,$$

$${}^{r)}\mathrm{D}^{n+c} + \frac{c}{c-1}\,.\,{}^{1}\varDelta^{c-1}\,.\,{}^{r-1)}\mathrm{D}^{n+c} + \frac{c}{c-2}\,.\,{}^{2}\varDelta^{c-2}\,.\,{}^{r-2)}\mathrm{D}^{n+c} + \ldots,$$

$$r\,.\,{}^{r)}\mathrm{D}^{n+c} - r\,\frac{r-1}{2}\,.\,{}^{r)}\mathrm{D}^{n+2c} + \ldots + \left(\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r-1}\,.\,{}^{1}\varDelta^{n+r-1}\,.\,{}^{r-1}\varDelta^{1-n-r} - \frac{n}{n+r-2}\,.\,{}^{2}\varDelta^{n+r-2}\,.\,{}^{r-2}\varDelta^{2-n-r} - \ldots,$$

$$-\frac{n}{n+r}\left({}^{1)}\mathrm{D}^{-n-r}\,.\,{}^{r-1}\varDelta^{1-n-r} + {}^{2)}\mathrm{D}^{-n-r}\,.\,{}^{r-2}\varDelta^{2-n-r} + \ldots\right),$$

$$n\,\frac{n-1}{n+r}\left(\frac{{}^{r}\Delta^{1-n-r}}{n+r-1} - \frac{{}^{r-1}\Delta^{2-n-r}}{n+r-2} + \frac{{}^{r-2}\Delta^{3-n-r}}{n+r-3} - \ldots\right),$$

$$\frac{c-n}{c-n-r}\,.\,{}^{r}\varDelta^{c-n-r} + \frac{c-n}{c-n-r+1}\,.\,{}^{1)}\mathrm{D}^{c}\,.\,{}^{r-1}\varDelta^{c-n-r+1} + \ldots,$$

$$\frac{c-n}{c-n-r}\,.\,{}^{r}\varDelta^{c-n-r} + \frac{c-n}{c-n-r+1}\,\frac{c}{c+1}\,.\,{}^{1}\varDelta^{-c-1}\,.\,{}^{r-1}\varDelta^{c-n-r+1} + \ldots,$$

$$r\,\frac{c-n}{c-n-r}\,.\,{}^{r}\varDelta^{c-n-r} - r\,\frac{r-1}{2}\,\frac{2c-n}{2c-n-r}\,.\,{}^{r}\varDelta^{2c-n-r} + \ldots + \left(-\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r}\left({}^{1)}\mathrm{D}^{-n-1}\,.\,{}^{r-1}\varDelta^{-n-r} + {}^{2)}\mathrm{D}^{-n-2}\,.\,{}^{r-2}\varDelta^{-n-r} + \ldots\right),$$

$$-\frac{n+1}{n+r}\,.\,{}^{1}\varDelta^{n}\,.\,{}^{r-1}\varDelta^{-n-r} - \frac{n+2}{n+r}\,.\,{}^{2}\varDelta^{n}\,.\,{}^{r-2}\varDelta^{-n-r} - \ldots,$$

$$\frac{-n}{2^{r}-1}\left(\frac{{}^{r-1}\Delta^{1-n-r}}{2} - \frac{n+r+1}{2}\,\frac{{}^{r-2}\Delta^{2-n-r}}{2^{2}} + \frac{n+r+1}{2}\,\frac{n+r+2}{3}\,\frac{{}^{r-3}\Delta^{3-n-r}}{2^{3}} - \ldots\right),$$

$${}^{r}\varDelta^{-n-r-1} + {}^{r-1}\varDelta^{-n-r-1} + \frac{{}^{r-2}\Delta^{-n-r-1}}{1.2} + \ldots,$$

$$\frac{c-n}{c-n-r}\,.\,{}^{r}\varDelta^{c-n-r} + \frac{c-n-1}{c-n-r}\,.\,{}^{1}\varDelta^{-c}\,.\,{}^{r-1}\varDelta^{c-n-r} + \ldots,$$

$$\frac{c-n}{c-n-r}\,.\,{}^{r}\varDelta^{c-n-r} + \frac{c-n-1}{c-n-r}\,\frac{c}{c-1}\,.\,{}^{1)}\mathrm{D}^{c-1}\,.\,{}^{r-1}\varDelta^{c-n-r} + \ldots.$$

TABLE OF ${}^r\Delta^n = \dfrac{\Delta^n 0^{n+r}}{\Gamma(n+r+1)}$ AND OF ${}^{r)}\mathrm{D}^n = \dfrac{\mathrm{D}^n 0^{n+r)}}{\Gamma(n+r+1)}$, ACCOMPANYING PARAGRAPH 95.

r	${}^r\Delta^{-6}$	${}^r\Delta^{-5}$	${}^r\Delta^{-4}$	${}^r\Delta^{-3}$	${}^r\Delta^{-2}$	${}^r\Delta^{-1}$	${}^r\Delta^{0}$	${}^r\Delta^{1}$	${}^r\Delta^{2}$	${}^r\Delta^{3}$	${}^r\Delta^{4}$	${}^r\Delta^{5}$	${}^r\Delta^{6}$	r
0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	1
2	$\frac{17}{4}$	$\frac{35}{12}$	$\frac{11}{6}$	1	$\frac{5}{12}$	$\frac{1}{12}$	0	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{5}{4}$	$\frac{13}{6}$	$\frac{10}{3}$	$\frac{19}{4}$	2
3	$-\frac{15}{4}$	$-\frac{25}{12}$	-1	$-\frac{3}{8}$	$-\frac{1}{12}$	0	0	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{25}{8}$	$\frac{21}{4}$	3
4	$\frac{137}{60}$	1	$\frac{251}{720}$	$\frac{19}{240}$	$\frac{1}{240}$	$\frac{-1}{720}$	0	$\frac{1}{120}$	$\frac{31}{360}$	$\frac{43}{120}$	$\frac{81}{80}$	$\frac{331}{144}$	$\frac{1087}{240}$	4
5	-1	$\frac{-95}{288}$	$-\frac{3}{40}$	$\frac{-1}{160}$	$\frac{1}{720}$	0	0	$\frac{1}{720}$	$\frac{1}{40}$	$\frac{23}{160}$	$\frac{37}{72}$	$\frac{45}{32}$	$\frac{259}{80}$	5
6	$\frac{19087}{60480}$	$\frac{863}{12096}$	$\frac{221}{30240}$	$\frac{-1}{945}$	$\frac{-1}{6048}$	$\frac{1}{30240}$	0	$\frac{1}{5040}$	$\frac{127}{20160}$	$\frac{605}{12096}$	$\frac{6821}{30240}$	$\frac{2243}{3024}$	$\frac{30083}{15120}$	6
7	$\frac{-275}{4032}$	$\frac{-95}{12096}$	$\frac{11}{15120}$	$\frac{1}{4032}$	$\frac{-1}{30240}$	0	0	$\frac{1}{40320}$	$\frac{17}{12096}$	$\frac{311}{20160}$	$\frac{265}{3024}$	$\frac{1045}{3024}$	$\frac{97}{90}$	7
8	$\frac{9829}{1209600}$	$\frac{-47}{103680}$	$\frac{-199}{725760}$	$\frac{19}{1209600}$	$\frac{1}{172800}$	$\frac{-1}{1209600}$	0	$\frac{1}{362880}$	$\frac{73}{259200}$	$\frac{2591}{604800}$	$\frac{55591}{1814400}$	$\frac{7501}{51840}$	$\frac{63373}{120960}$	8
9	$\frac{19}{80640}$	$\frac{79}{290304}$	$\frac{1}{604800}$	$\frac{-1}{115200}$	$\frac{1}{1209600}$	0	0	$\frac{1}{3628800}$	$\frac{31}{604800}$	$\frac{437}{403200}$	$\frac{253}{25920}$	$\frac{2669}{48384}$	$\frac{6671}{28800}$	9

r	${}^{r)}\mathrm{D}^{-6}$	${}^{r)}\mathrm{D}^{-5}$	${}^{r)}\mathrm{D}^{-4}$	${}^{r)}\mathrm{D}^{-3}$	${}^{r)}\mathrm{D}^{-2}$	${}^{r)}\mathrm{D}^{-1}$	${}^{r)}\mathrm{D}^{0}$	${}^{r)}\mathrm{D}^{1}$	${}^{r)}\mathrm{D}^{2}$	${}^{r)}\mathrm{D}^{3}$	${}^{r)}\mathrm{D}^{4}$	${}^{r)}\mathrm{D}^{5}$	${}^{r)}\mathrm{D}^{6}$	r
0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3	1
2	$\frac{13}{4}$	$\frac{25}{12}$	$\frac{7}{6}$	$\frac{1}{2}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{11}{12}$	$\frac{7}{4}$	$\frac{17}{6}$	$\frac{25}{6}$	$\frac{23}{4}$	2
3	$\frac{3}{2}$	$\frac{5}{8}$	$\frac{1}{6}$	0	0	$\frac{1}{24}$	0	$-\frac{1}{4}$	$-\frac{5}{6}$	$-\frac{15}{8}$	$-\frac{7}{2}$	$-\frac{35}{6}$	-9	3
4	$\frac{31}{120}$	$\frac{1}{24}$	$\frac{-1}{720}$	$\frac{1}{240}$	$\frac{-1}{240}$	$\frac{-19}{720}$	0	$\frac{1}{5}$	$\frac{137}{180}$	$\frac{29}{15}$	$\frac{967}{240}$	$\frac{1069}{144}$	$\frac{3013}{240}$	4
5	$\frac{1}{120}$	0	0	$\frac{-1}{480}$	$\frac{1}{240}$	$\frac{3}{160}$	0	$-\frac{1}{6}$	$-\frac{7}{10}$	$-\frac{469}{240}$	$-\frac{89}{20}$	$-\frac{285}{32}$	$-\frac{781}{48}$	5
6	$\frac{1}{30240}$	$\frac{-1}{6048}$	$\frac{1}{3024}$	$\frac{1}{945}$	$\frac{-221}{60480}$	$\frac{-863}{60480}$	0	$\frac{1}{7}$	$\frac{363}{560}$	$\frac{29531}{15120}$	$\frac{4523}{945}$	$\frac{31063}{3024}$	$\frac{242537}{12096}$	6
7	0	$\frac{1}{12096}$	$\frac{-1}{3024}$	$\frac{-11}{20160}$	$\frac{19}{6048}$	$\frac{275}{24192}$	0	$-\frac{1}{8}$	$\frac{-761}{1260}$	$-\frac{1303}{672}$	$-\frac{7645}{1512}$	$-\frac{139381}{12096}$	$-\frac{48035}{2016}$	7
8	$\frac{-1}{57600}$	$\frac{-19}{725760}$	$\frac{199}{725760}$	$\frac{47}{172800}$	$\frac{-9829}{3628800}$	$\frac{-33953}{3628800}$	0	$\frac{1}{9}$	$\frac{7129}{12600}$	$\frac{16103}{8400}$	$\frac{341747}{64800}$	$\frac{1148963}{90720}$	$\frac{1666393}{60480}$	8
9	$\frac{1}{57600}$	$\frac{-1}{483840}$	$\frac{-79}{362880}$	$\frac{-19}{161280}$	$\frac{407}{172800}$	$\frac{8183}{1036800}$	0	$-\frac{1}{10}$	$\frac{-671}{1260}$	$-\frac{190553}{100800}$	$-\frac{412009}{75600}$	$-\frac{355277}{25920}$	$-\frac{22463}{720}$	9

96. The factorial theorem may be applied in many cases not contemplated in the foregoing analysis, cases the discussion of which requires the consideration of a class of functions much wider than that already widened class to which I have extended the name of factorial. The chief mark of a factorial, as I have defined it, is the property (266), $\partial x^{[m]} = mx^{[m-1]}$. That this is not the only mark, however, is readily to be seen, since any function of x and m might be made the starting point of a series of functions possessed of that property, to be developed by performing the operation ∂ or ∂^{-1}, it being understood that $m^{-1}\partial\phi(x, m)$ must be called $\phi(x, m-1)$. The form of the function, however, would, in most cases, be liable to perpetual alteration, and it is only such functions as retain their form after being operated upon by ∂ that can to advantage be classed together under the same name.

97. I call $x^{[m]}$ a primary factorial because it is the simplest form of function of which we may say $\partial\phi(x, m) = m\phi(x, m-1)$. If we make $x^{[0]} = 1$, which is clearly its simplest form, we find that

$$x^{[1]} = \partial^{-1}x^{[0]} = x. \tag{347}$$

The possible complementary constant is, for simplicity, disregarded. Similarly,

$$x^{[2]} = 2\partial^{-1}x = x(x + 2ah - h); \tag{348}$$

and by repeated operation we find that (267) is the simplest general form of function to which the property $\partial\phi(x, m) = m\phi(x, m-1)$ can be ascribed.

98. Let Q be any function of E; then, whatever value be assigned to m,

$$\partial \text{Q} x^{[m]} = \text{Q}\partial x^{[m]} = \text{Q}mx^{[m-1]} = m\text{Q}x^{[m-1]}. \tag{349}$$

Let $\text{Q}x^{[m]}$ be represented by $x^{\{m\}}$; then

$$\partial x^{\{m\}} = mx^{\{m-1\}}. \tag{350}$$

In $x^{\{m\}}$ we have, I think, the most general form of function to which the term factorial can conveniently be applied.

99. We can now extend widely the applicability of the factorial theorem. Let $\text{G} = \text{Q}^{-1}$, where Q is any function of E to which that theorem applies. Then, writing ϕEQ for ϕE in (291),

$$\phi\text{EG}^{-1} = \phi\text{EQ} = \phi\text{EQ}0^{[0]} + \phi\text{EQ}0^{[1]}.\partial + \ldots. \tag{351}$$

Writing $0^{\{m\}}$ for $\text{Q}0^{[m]}$, and multiplying both sides by G, we have the factorial theorem in this more general form,

$$\phi\text{E} = \phi\text{E}0^{\{0\}}.\text{G} + \phi\text{E}0^{\{1\}}.\text{G}\partial + \frac{1}{2}\phi\text{E}0^{\{2\}}.\text{G}\partial^2 + \ldots. \tag{352}$$

From this may be drawn deductions similar, *mutatis mutandis*, to those already made from the factorial theorem.

100. Of the forms which G can assume, one of the most important is

$$\mathrm{G} = a\mathrm{E}^{-ah} + (1-a)\,\mathrm{E}^{(1-a)h}. \tag{353}$$

In this case, let what $x^{\{m\}}$ becomes be denoted by $x^{]m[}$, employing reversed brackets. Then, for all values of m,

$$x^{]m[} = x^{-1}x^{[m+1]}, \tag{354}$$

as may be shown by performing the operation G on both sides, resulting in the normal equation $\mathrm{G}x^{]m[} = x^{[m]}$. In general, therefore,

$$x^{]m[} = h^m\,\frac{\Gamma(xh^{-1} + a\,[m+1])}{\Gamma(xh^{-1} + a\,[m+1] - m)}, \tag{355}$$

and when m is positive and integral,

$$x^{]m[} = (x + a\,[m+1]\,h - h)(x + a\,[m+1]\,h - 2h)\ldots(x + [a-1][m+1]\,h + h). \tag{356}$$

Let us call all functions of this form Secondary Factorials. Just as we have distinguished certain functions as primary factorials, because they are the simplest functions complying with the conditions $\partial\phi\,(x, m) = m\phi(x, m-1)$ and $\phi\,(x, 0) = x^0$, so we may remark in this case that the class of secondary factorials comprises those which comply with the conditions $\partial\phi\,(x, m) = m\phi\,(x, m-1)$ and $\phi\,(x, -1) = x^{-1}$.

101. The investigation of negative factorials is not required in connection with the object for which I have embraced the theory of factorials in this essay, namely, the development and illustration of the factorial theorem. I shall, therefore, dismiss that branch of the subject with but transient consideration. When m is a negative integer, we have, as the general form of primary factorials,

$$x^{[m]} = \frac{x}{(x + amh)(x + amh - h)\ldots(x + [a-1]\,mh)}; \tag{357}$$

and, when m is any negative integer (except -1, when $x^{]-1[} = x^{-1}$),

$$x^{]m[} = \frac{1}{(x + a\,[m+1]\,h)(x + a\,[m+1]\,h - h)\ldots(x + [a-1]\,[m+1]\,h)} \tag{358}$$

as the general form of secondary factorials. By repeated operation,

$$\partial^n x^{]-1[\prime} = \partial^n x^{]-1[} = (-1)^n\,\Gamma(1+n)\,x^{]-n-1[}. \tag{359}$$

But, by (338),

$$\begin{aligned}\partial^n x^{]-1[\prime} &= (\partial'^n + {}^{[1]\prime}\partial^n\,.\,\partial'^{n+1} + \ldots)\,x^{]-1[\prime} \\ &= (-1)^n\,\Gamma(1+n)\,(x^{]-n-1[\prime} - [n+1]\,.\,{}^{[1]\prime}\partial^n\,.\,x^{]-n-2[\prime} + \ldots)\,.\end{aligned} \tag{360}$$

Comparing these two expressions, and writing $n-1$ for n, we have this general theorem for the transformation of one form of negative integral secondary factorial into another,

$$x^{]-n[} = x^{]-n[\prime} - n\,.\,{}^{[1]\prime}\partial^{n-1}\,.\,x^{]-n-1[\prime} + n\,(n+1)\,.\,{}^{[2]\prime}\partial^{n-1}\,.\,x^{]-n-2[\prime} - \ldots. \tag{361}$$

Since $x^{[-n]} = x x^{]-n-1[}$, we obtain also the following general theorem for the transformation of negative integral primary factorials,

$$x^{[-n]} = x^{[-n]\prime} - (n+1)\,.^{[1]\prime}\partial^n.\,x^{[-n-1]\prime} + (n+1)(n+2)\,.^{[2]\prime}\partial^n.\,x^{[-n-2]\prime} - \ldots. \qquad (362)$$

102. We may distinguish the principal special cases of secondary, and also negative, factorials in a manner similar to that employed for the distinction of the corresponding special cases of positive primary factorials. The following table, showing the chief special forms of $x^{[3]}$ and $x^{[-3]}$, $x^{]3[}$ and $x^{]-3[}$, will afford a sufficient illustration of the use of the various special symbols which represent the most important varieties of factorials.

		h	a	PRIMARY.	SECONDARY.
Positive:	Power	0		$x^3 = xxx$	$x^3 = xxx$
	Upper	1	0	$x^{3)} = x(x-1)(x-2)$	$x^{3(} = (x-1)(x-2)(x-3)$
	Lower	1	1	$x^{(3} = x(x+2)(x+1)$	$x^{)3} = (x+3)(x+2)(x+1)$
	Central	1	$\frac{1}{2}$	$x^{(3)} = x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$	$x^{)3(} = (x+1)\,x\,(x-1)$
Negative:	Power	0		$x^{-3} = \frac{1}{xxx}$	$x^{-3} = \frac{1}{xxx}$
	Upper	1	0	$x^{-3)} = \frac{1}{(x+1)(x+2)(x+3)}$	$x^{-3(} = \frac{1}{x(x+1)(x+2)}$
	Lower	1	1	$x^{(-3} = \frac{1}{(x-3)(x-2)(x-1)}$	$x^{)-3} = \frac{1}{(x-2)(x-1)\,x}$
	Central	1	$\frac{1}{2}$	$x^{(-3)} = \frac{x}{\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)}$	$x^{)-3(} = \frac{1}{(x-1)\,x\,(x+1)}$

Exclusive of x^m, the most important forms of $x^{[m]}$ are $x^{m)}$, $x^{(m)}$, and $x^{)m(}$, corresponding respectively to the three most important forms of difference, Δ, Λ and $\text{I}\Lambda$. Lower differences and lower factorials are comparatively unimportant, since $\phi\,\grave{}\Delta\psi x = \phi(-\Delta)\,\psi(-y)$, where $y = -x$, a slight transformation thus enabling us to use Δ instead of $\grave{}\Delta$.

103. For expansion in terms of mean central differences, we may, in (352), write I for G and $0^{)m(}$ for $0^{\{m\}}$, when the factorial theorem assumes this shape:

$$\phi\text{E} = \phi\text{E}0^{)0(}.\,\text{I} + \phi\text{E}0^{)1(}.\,\text{I}\Lambda + \frac{1}{2}\,\phi\text{E}0^{)2(}.\,\text{I}\Lambda^2 + \ldots. \qquad (363)$$

To illustrate this formula, let $\phi\text{E} = \text{E}^0$, and we have at once the well known formula for interpolation, due to Gauss, and now assuming the following symmetrical form:

$$\text{E}^0 = \text{I} + \frac{0^{)2(}}{2}\,\text{I}\Lambda^2 + \frac{0^{)4(}}{2\,.\,3\,.\,4}\,\text{I}\Lambda^4 + \ldots. \qquad (364)$$

Again, let $\phi\text{E} = \text{D}$; then

$$\text{D} = \text{I}\Lambda + \frac{\text{D}0^{)3(}}{2\,.\,3}\,\text{I}\Lambda^3 + \frac{\text{D}0^{)5(}}{2\,.\,3\,.\,4\,.\,5}\,\text{I}\Lambda^5 + \ldots. \qquad (365)$$

Thus is disclosed, in the neatest possible form, the law of the known series (256). Another special case of (363) is the following important, and probably new, summation formula:

$$\frac{1}{1+\mathrm{E}} = \frac{1}{2} - \frac{1}{4}\,\mathrm{I}\Delta + \frac{1}{4^2}\,\mathrm{I}\Delta^3 - \frac{1}{4^3}\,\mathrm{I}\Delta^5 + \ldots . \tag{366}$$

This formula, which is otherwise easily demonstrable, since $\frac{1}{1+\mathrm{E}} = \frac{1}{2} - \frac{\mathrm{I}\Delta}{4+\Delta^2}$, may be interpreted as follows. If $\ldots a_{-1},\ a_0,\ a_1, \ldots$ be any series,

$$a_0 - a_1 + a_2 - \ldots = \frac{1}{2}\,a_0 - \frac{1}{4}\,\mathrm{I}\Delta a_0 + \frac{1}{4^2}\,\mathrm{I}\Delta^3 a_0 - \ldots . \tag{367}$$

This theorem will doubtless be found at least equal in usefulness to the corresponding formula in upper differences, and superior to Hutton's method of summing quantities alternately positive and negative. It is almost needless to say that $\mathrm{I}\Delta^n$ may be written in (332) for ∂^n, or, if $0^{)m(}$ is also written for $0^{[m]}$, for ∂^n, according as it is desired to transform mean central differences, positive or negative, into other forms of difference-ratio, or *vice versa.* It does not seem necessary, for present purposes, to enter into a recital of the proof of this statement. The coefficients represented by ${}^r\mathrm{I}\Delta^n$ and ${}^{)r(}\mathrm{D}^n$ are of considerable importance, and should be tabulated.

VII. *Theory of the Calculus of Multiplication.*

104. The Calculus of Enlargement is, as we have seen, based on that operation which changes ϕx into $\phi\,(x+h)$ by adding to the variable. If we seek the most simple repetitive operation which shall have the effect of multiplying x, instead of adding to it, we shall find that it consists in changing ϕx into $\phi\,(x\varepsilon^h)$. Let us denote this operation, as in paragraph 34, by the symbol M, so that

$$\mathrm{M}^h\phi x = \phi\,(x\varepsilon^h)\,. \tag{368}$$

The operation E^h, the basis of the Calculus of Enlargement, changes x in arithmetical ratio, so to speak, while the operation M^h, the basis of what we may call the Calculus of Multiplication, changes it in geometrical ratio.

105. We have seen that in this case $\mathrm{M} = \mathrm{S}$ where $\psi x = \log x$, and that all the results derivable from a possible Calculus of Multiplication can be obtained at once from the Calculus of Enlargement by expressing functions of x as functions of $u = \log x$, and observing that $\mathrm{M}_x = \mathrm{E}_u$. It is, therefore, unnecessary for any practical purpose to discuss that possible calculus.

Nevertheless such a discussion will not now be useless, for it will serve to illustrate and impress upon the mind the truth that the Calculus of Enlargement is not the only possible calculus, but is rather to be regarded as the simplest of many possible systems.

106. Let log M be represented by L; then, by (69),

$$\mathrm{M}^h = 1 + h\mathrm{L} + \frac{1}{2} h^2\mathrm{L}^2 + \dots, \tag{369}$$

$$\phi\,(x\varepsilon^h) = \phi x + h\mathrm{L}\phi x + \frac{1}{2} h^2\mathrm{L}^2\phi x + \dots. \tag{370}$$

Putting $x = 1$, and afterwards writing ϕ log for ϕ, we have

$$\phi\,(\varepsilon^h) = \phi 1 + h\mathrm{L}\phi 1 + \frac{1}{2} h^2\mathrm{L}^2\phi 1 + \dots, \tag{371}$$

$$\phi h = \phi 0 + h\mathrm{L}\phi \log 1 + \frac{1}{2} h^2\mathrm{L}^2\phi \log 1 + \dots. \tag{372}$$

These three theorems are respectively analogous to those of Taylor, Herschel, and Maclaurin. Assuming, what will be proved, that

$$\phi\mathrm{L}\psi \log 1 = \psi\mathrm{L}\phi \log 1, \tag{373}$$

we have also

$$\phi\,(\varepsilon^h) = \phi 1 + h\phi\mathrm{M} \log 1 + \frac{1}{2} h^2\phi\mathrm{M}\,(\log 1)^2 + \dots, \tag{374}$$

$$\phi h = \phi 0 + h\phi\mathrm{L} \log 1 + \frac{1}{2} h^2\phi\mathrm{L}\,(\log 1)^2 + \dots. \tag{375}$$

107. From (75),

$$\mathrm{L} \log x = 1. \tag{376}$$

From (77) and (78),

$$\phi\mathrm{M}_x\psi\,(u, v, w, \dots) = \phi\,(\mathrm{M}_{x\,|\,u}\mathrm{M}_{x\,|\,v}\mathrm{M}_{x\,|\,w}\dots)\,\psi\,(u, v, w, \dots), \tag{377}$$

$$\phi\mathrm{M}_{x\,|\,u}\psi\,(u, v) = \phi\,(\mathrm{M}_x\mathrm{M}_{x\,|\,v}{}^{-1})\,\psi\,(u, v). \tag{378}$$

From (88),

$$\mathrm{L}_x v = \mathrm{L}_x \log u \,.\, \mathrm{L}_u v. \tag{379}$$

108. By comparison of the general terms of the expansions of the two members in each case, respectively, we find that

$$\phi\mathrm{M}_x\psi\,(xy) = \phi\mathrm{M}_y\psi\,(xy), \tag{380}$$

$$\phi\,(y\mathrm{M}_x)\,\psi x = \psi\,(x\mathrm{M}_y)\,\phi y, \tag{381}$$

$$\phi\mathrm{M}x^m\psi x = x^m\phi\,(\varepsilon^m\mathrm{M})\,\psi x, \tag{382}$$

$$\phi\,(\mathrm{M}_x\mathrm{M}_y\dots)\,s = s\phi\,(\varepsilon^n), \tag{383}$$

where s is a quantic of the nth degree, say $s = x^hy^k\dots$, where $h + k + \dots = n$; also

$$\phi\,(\mathrm{M}_x\mathrm{M}_y\dots)\,s\psi\,(x, y, \dots) = s\phi\,(\varepsilon^n\mathrm{M}_x\mathrm{M}_y\dots)\,\psi\,(x, y, \dots). \tag{384}$$

If, in (381), $y = 1$,

$$\phi\mathrm{M}\psi x = \psi\,(x\mathrm{M}_1)\,\phi 1, \tag{385}$$

$$\phi M x^m = x^m M_1^m \phi 1 = x^m \phi(\varepsilon^m), \tag{386}$$

$$\phi M_x c = c\phi 1, \tag{387}$$

c being anything independent of x.

109. Writing, in the foregoing formulæ, ϕ log for ϕ, we have these general theorems,

$$\phi L_x \psi(u, v, w, \ldots) = \phi(L_{x|u} + L_{x|v} + L_{x|w} + \ldots)\, \psi(u, v, w, \ldots), \tag{388}$$

$$\phi L_{x|u}\psi(u, v) = \phi(L_x - L_{x|v})\, \psi(u, v), \tag{389}$$

$$\phi L_x \psi(xy) = \phi L_y \psi(xy), \tag{390}$$

$$\phi(\log y + L_x)\, \psi x = \psi(x M_y)\, \phi \log y, \tag{391}$$

$$\phi L x^m \psi x = x^m \phi(m + L)\, \psi x, \tag{392}$$

$$\phi(L_x + L_y + \ldots)\, s = s\phi n, \tag{393}$$

$$\phi(L_x + L_y + \ldots)\, s\psi(x, y, \ldots) = s\phi(n + L_x + L_y + \ldots)\, \psi(x, y, \ldots), \tag{394}$$

$$\phi L \psi x = \psi(x M_1)\, \phi \log 1, \tag{395}$$

$$\phi L x^m = x^m \phi m, \tag{396}$$

$$\phi L c = c\phi 0. \tag{397}$$

From (395), writing ψ log for ψ, and putting $x = 1$, we have (373). As special cases of some of these general theorems, we obtain

$$L_x^n \psi(u, v, \ldots) = (L_{x|u} + L_{x|v} + \ldots)^n \psi(u, v, \ldots), \tag{398}$$

$$L_x uv = u L_x v + v L_x u, \tag{399}$$

$$L_{x|u}{}^n \psi(u, v) = (L_x - L_{x|v})^n \psi(u, v), \tag{400}$$

$$L^n x^m = x^m m^n, \tag{401}$$

$$L x^m = m x^m, \tag{402}$$

$$L c = 0. \tag{403}$$

110. From (379), since $L_u u = u$,

$$L_x u = u L_x \log u, \tag{404}$$

$$L \varepsilon^{\phi x} = \varepsilon^{\phi x} L \phi x, \tag{405}$$

$$L \varepsilon^{kx} = \varepsilon^{kx} L kx = \varepsilon^{kx} kx. \tag{406}$$

To illustrate (404),

$$L x^m = x^m L m \log x = m x^m, \tag{407}$$

$$L_x \log u = \frac{L_x u}{u}, \tag{408}$$

$$L \log x = \frac{Lx}{x} = 1. \tag{409}$$

111. If we know the development of $\phi(x\varepsilon^h)$ in positive integral powers of h, we can tell from it the value of $L\phi x$, which, by (370), is the coefficient of h in such development. Thus, since $x^m \varepsilon^{hm} = x^m(1 + hm + \ldots)$, we have $L x^m = m x^m$, and since $\log(x\varepsilon^h) = \log x + h$, we have $L \log x = 1$. Again, the

various special cases of the general logarithmic series (31) will afford equivalents for L in terms of M or of simple functions of M, by means of which we may ascertain $\text{L}\phi x$. It is needless to recite these cases. As illustrations,

$$\text{L}x^m = \frac{\text{M}^0 - 1}{0}\, x^m = x^m\, \frac{\varepsilon^{m0} - 1}{0} = mx^m, \tag{410}$$

$$\text{L} \log x = \frac{\log (x\varepsilon^0) - \log x}{0} = 1, \tag{411}$$

$$\text{L}x^m = (\text{M} - 1)\, x^m - \frac{1}{2}\, (\text{M} - 1)^2 x^m + \ldots = x^m([\varepsilon^m - 1] - \ldots) = mx^m. \tag{412}$$

112. It follows from (403) that, when the inverse operation L^{-1} is performed, a complementary constant must be introduced. Performing that operation on (399) we have

$$\text{L}_x^{-1} u\text{L}_x v = uv - \text{L}_x^{-1} v\text{L}_x u\,. \tag{413}$$

For example,

$$\begin{aligned} \text{L}^{-1} x\varepsilon^x &= x\varepsilon^x - \text{L}^{-1} x^2\varepsilon^x \\ &= x\varepsilon^x - \frac{1}{2}\, x^2\varepsilon^x + \text{L}^{-1}\, \frac{1}{2}\, x^3\varepsilon^x \\ &= x\varepsilon^x - \frac{1}{2}\, x^2\varepsilon^x + \frac{1}{2\,.\,3}\, x^3\varepsilon^x - \ldots + c\,. \end{aligned} \tag{414}$$

But $\text{L}^{-1} x\varepsilon^x = \varepsilon^x$. Substituting this in (414), dividing throughout by ε^x, and writing $-x$ for x, we have

$$c\varepsilon^x = 1 + x + \frac{1}{2}\, x^2 + \ldots, \tag{415}$$

wherein putting $x = 0$ shows that $c = 1$.

113. If ϕx is algebraically less than $\phi\, (x\varepsilon^h)$ for all values of h lying between some positive quantity and some other negative quantity, exclusive of the value $h = 0$, it is a minimum, and if greater than $\phi\, (x\varepsilon^h)$ for all such values, a maximum. If, in

$$\phi\, (x\varepsilon^h) = \phi x + h\text{L}\phi x + \frac{1}{2}\, h^2\text{L}^2\phi x + \ldots, \tag{416}$$

$\text{L}\phi x$ does not vanish, ϕx is neither a maximum nor a minimum, unless, indeed, $\text{L}\phi x$ is infinite, a case which we need not now consider; for, by making h small enough to cause $h\text{L}\phi x$ to exceed the sum of all succeeding terms, $\phi\, (x\varepsilon^h) - \phi x$ and $\phi\, (x\varepsilon^{-h}) - \phi x$ will have different signs. If $h\text{L}\phi x = 0$,

$$\phi\, (x\varepsilon^h) = \phi x + \frac{1}{2}\, h^2\text{L}^2\phi x + \ldots, \tag{417}$$

and when h is small enough to cause $\frac{1}{2}\, h^2\text{L}^2\phi x$ to exceed the sum of all succeeding terms, $\phi\, (x\varepsilon^h) - \phi x$ and $\phi\, (x\varepsilon^{-h}) - \phi x$ will have the same sign, and ϕx

will be a maximum or a minimum, provided $L^2\phi x$ does not vanish, in which case the matter is still in doubt. If $L^2\phi x$ is negative, ϕx is a maximum, and *vice versa.* It may in this manner be shown that for ϕx to be a maximum or minimum an odd number of powers of h must vanish, in which case the coefficient of the next succeeding power will, by its sign, determine whether ϕx is a maximum or a minimum. For example, let $\phi x = \varepsilon^{1-x} + \varepsilon^{x-1} + 2\cos(x-1)$. In this case it will be found that, when $x = 1$, $L\phi x$, $L^2\phi x$ and $L^3\phi x$ all vanish, and $L^4\phi x = 4$, showing that ϕx is a minimum.

114. Since the processes of any calculus may be expressed in the language of any other calculus, those of the Calculus of Enlargement may be expressed in the language of the Calculus of Multiplication. For example, let us in (370) write ϕ log for ϕ and x for log x; then, if $u = \varepsilon^x$,

$$\phi(x+h) = \phi x + hL_u\phi x + \frac{1}{2}h^2L_u^2\phi x + \ldots, \tag{418}$$

a form of Taylor's theorem. Since by (379) $L_u = x^{-1}L_x$,

$$\phi(x+h) = \phi x + hx^{-1}L_x\phi x + \frac{1}{2}h^2(x^{-1}L_x)^2\phi x + \ldots. \tag{419}$$

The results of the Calculus of Enlargement may thus, in general, be obtained by the processes of the Calculus of Multiplication; though the former method must, of course, be preferred on the ground of simplicity. For expressing the results of the Calculus of Multiplication in the language of the Calculus of Enlargement we have, from (93),

$$L_x = L_x x \,.\, D_x = xD_x\,. \tag{420}$$

C. Summary.

115. The Calculus of Enlargement relates to the theory and practice of certain operations, embracing in its field the Calculus of Finite Differences, the Differential and Integral Calculus, and the Calculus of Variations. The operations comprised by it are those whose symbols are functions of E, the symbol of Enlargement, the operation by which ϕx becomes $\phi(x+1)$. It would be possible to form any number of systems, each a calculus comprising operations whose symbols are functions of some simple symbol other than E, but the results so obtainable can be got from the Calculus of Enlargement, and the elaboration of such other systems would, therefore, be superfluous.

116. All functions of E may be treated separately from the subject of operation, by any algebraic rules applicable to symbols in general, the theory

of the functions of E forming an Algebra of which the theory of Differentiation is that part corresponding to the theory of Logarithms in ordinary Algebra. The symbol of Differentiation, D, is the logarithm of E, the symbol of Enlargement. Whatever theorems may be proved regarding functions of E, as such, are true of D as one such function. It is worth remarking that at first the theory of logarithms was treated in a far-fetched and comparatively obscure manner, in connection with the properties of the hyperbola; many years passing before it was reduced to a simpler form as a branch of algebra. It is not without historical analogy, therefore, that the doctrine of differentiation has hitherto failed to find its true place as the logarithmic branch of that wider algebra, the doctrine of the functions of E, or Calculus of Enlargement.*

117. For the direct interpretation of D it is necessary to write E for x in expressions giving log x in terms of x or of simple functions of x. The two general logarithmic theorems (30) and (31) embrace, as special cases, many such expressions. For the better understanding of logarithms, it is best to refrain, in the definition of a logarithm, from describing it as an exponent. To explain differentials, we have the definition $d = \log e$, where e represents enlargement with respect to a hypothetical variable; and also, for infinitesimal differentials, the equation $d = \mathrm{E}^{(1-\alpha)0} - \mathrm{E}^{-\alpha 0}$. The most important expressions equivalent to D are three principal vanishing fractions and three corresponding series. The symbolic vanishing fractions appear to be novel, though the practical application of one or two of them is familiar; while, on the other hand, the series are well known in their symbolic form, though their practical use as definitions of D does not seem to have been previously suggested. These fractions and series are special cases of a single differentiate-expression (220), which is itself a special case of the factorial theorem. Taken all together, and in connection with other equivalents of D, they convey a broad and comprehensive idea of the meaning of the operation of differentiation; the notion afforded by the vanishing fractions being at once the hardest to grasp and the most satisfactory when clearly understood.

*"In fact, the arrangement of the truths of analytical science, such as history gives it, is very different from their logical and natural arrangement; and as, in the infancy of analysis, mathematicians were more solicitous to advance it, than to advance it by just and natural means, they frequently deviated into indirect and foreign demonstrations. . . . The evil attending on this mode of procedure has been, . . . that the principles of a general method have been sought for in some particular method, properly (that is, according to the logical and natural order of ideas) to be comprehended under the general one." Woodhouse, *Analytical Calculations*, p. 40. This was written relative to the history of log x, but applies equally well to that of log E.

The Motion of a Solid in a Fluid.

By Thomas Craig, *Fellow of Johns Hopkins University.*

In the following paper I have given a brief account of some of the most important work that has been done upon this problem, together with some additions to the theory which I believe to be new. The method that I have given of transformation by means of elliptic coordinates seems to me to be very simple and practical, depending, as it does, upon the most elementary properties of the coefficients in a certain system of linear equations.

The fluid under consideration is assumed to be perfect, incompressible and extending to infinity in all directions; and further, the space occupied by the fluid is supposed simply-connected and consequently the velocity potential single-valued. A velocity potential will exist; as we assume that the fluid is originally at rest, and that the entire motion of the system is due to the motion of the solid, and in consequence there will be no rotational motion generated among the fluid particles.

Designate by u, v, w the component velocities of translation of a point in the body with respect to a set of rectangular axes x, y, z fixed in the body, and by p, q, r the component angular velocities of the body around these axes. Now, letting n denote the outer normal to the surface of the body, we have for the determination of the velocity potential ϕ the equation

1. $$\frac{\partial\varphi}{\partial n}=(u+zq-yr)\cos(n,x)+(v+xr-zp)\cos(n,y)+(w+yp-xq)\cos(n,z).$$

Since the fluid is to be at rest at infinity, the first derivatives of ϕ with respect to x, y and z will vanish for infinitely great values of these variables; and since $\Delta^2\phi=0$ throughout the entire space, and ϕ with its derivatives is single-valued and continuous, we can write

2. $$\phi=u\phi_1+v\phi_2+w\phi_3+p\phi_4+q\phi_5+r\phi_6,$$

a linear equation in the six quantities u, v, &c. The six functions ϕ_1, ϕ_2, &c., satisfy the equation $\Delta^2\phi=0$, and, at the surface of the body, the relations

3. $$\begin{aligned}\frac{\partial\varphi_1}{\partial n}&=\cos(n,x), & \frac{\partial\varphi_4}{\partial n}&=y\cos(n,z)-z\cos(n,y),\\ \frac{\partial\varphi_2}{\partial n}&=\cos(n,y), & \frac{\partial\varphi_5}{\partial n}&=z\cos(n,x)-x\cos(n,z),\\ \frac{\partial\varphi_3}{\partial n}&=\cos(n,z), & \frac{\partial\varphi_6}{\partial n}&=x\cos(n,y)-y\cos(n,x),\end{aligned}$$

The entire motion being due to the motion of the solid, we know that the energy of the system will be a quadratic function of the six quantities u, v, w, p, q, r. Denote the energy by T and we have,

$$
\begin{aligned}
2T = {} & a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + 2a_{12}uv + 2a_{13}uw + 2a_{23}vw \\
& + a_{44}p^2 + a_{55}q^2 + a_{66}r^2 + 2a_{45}pq + 2a_{46}pr + 2a_{56}qr \\
& + 2p\,[a_{14}u + a_{24}v + a_{34}w] \\
& + 2q\,[a_{15}u + a_{25}v + a_{35}w] \\
& + 2r\,[a_{16}u + a_{26}v + a_{36}w],
\end{aligned} \tag{4}
$$

the coefficients a_{ij} being constants depending upon the shape of the body and the distribution of mass in its interior. If we divide the energy T into two parts, T' and T'', the former may denote that portion of the entire energy due to the fluid—the latter, that to the body; then the coefficients a_{ij} are also divided into two parts and we may write $a_{ij} = a'_{ij} + a''_{ij}$. Kirchhoff has shown that

$$a'_{ij} = -\rho\int ds\phi_i \frac{\partial\varphi_j}{\partial n} = -\rho\int ds\phi_j \frac{\partial\varphi_i}{\partial n}.$$

Of course i and j have values from 1 to 6 inclusive. The values of the coefficients a''_{ij} are easily obtained from the expression for the energy of the solid, or

$$
\begin{aligned}
2T'' = \int dm\, \{ & (u^2 + v^2 + w^2) + (y^2 + z^2)\,p^2 + (x^2 + z^2)\,q^2 + (x^2 + y^2)\,r^2 \\
& + 2x\,(vr - wq) + 2y\,(wp - ur) + 2z\,(uq - vp) \\
& - 2yzqr - 2zxrp - 2xypq \}.
\end{aligned}
$$

We will now take up the Kirchhoffian equations of the motion of the solid, and for brevity write

$$
\begin{aligned}
U &= \frac{\partial T}{\partial u}, & P &= \frac{\partial T}{\partial p}, \\
V &= \frac{\partial T}{\partial v}, & Q &= \frac{\partial T}{\partial q}, \\
W &= \frac{\partial T}{\partial w}, & R &= \frac{\partial T}{\partial r}.
\end{aligned} \tag{5}
$$

These equations are then

$$
\begin{aligned}
\frac{dU}{dt} &= rV - qW, & \frac{dP}{dt} &= wV - vW + rQ - qR, \\
\frac{dV}{dt} &= pW - rU, & \frac{dQ}{dt} &= uW - wU + pR - rP, \\
\frac{dW}{dt} &= qU - pV, & \frac{dR}{dt} &= vU - uV + qP - pQ.
\end{aligned} \tag{6}
$$

Concerning the forces that act upon the body, we know that, whatever be the motion at any instant, we can conceive it generated instantaneously from

rest by a properly chosen impulse applied to the body; this impulse, according to the method of Poinsot, consisting of a force and a couple, whose axis is in the direction of the force. The quantities U, V, &c., are then the components with respect to the axes x, y, z of this force and couple; and the above equations show that these quantities vary only with the motion of the axes to which they are referred. Kirchhoff has observed that a particular solution of the above equations is obtained by supposing u, v, w constant and $p = q = r = 0$, provided we have

7. $$\frac{U}{u} = \frac{V}{v} = \frac{W}{w},$$

or,

7'. $$\frac{a_{11}u + a_{12}v + a_{13}w}{u} = \frac{a_{21}u + a_{22}v + a_{23}w}{v} = \frac{a_{31}u + a_{32}v + a_{33}w}{w};$$

(of course $a_{ij} = a_{ji}$), that is, provided the velocity, of which u, v, w are the rectangular components, be parallel to one of the principal axes of the ellipsoid,

8. $$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = \text{const.}$$

Calling λ the common value of the above quantities, we have for the determination of λ the cubic,

9. $$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & a_{13}, \\ a_{21}, & a_{22} - \lambda, & a_{23}, \\ a_{31}, & a_{32}, & a_{33} - \lambda, \end{vmatrix} = 0.$$

The eliminant of the equations of motion is

$$\begin{vmatrix} 0, & 0, & 0, & 0, & -W, & V, \\ 0, & 0, & 0, & W, & 0, & -U, \\ 0, & 0, & 0, & -V, & U, & 0, \\ 0, & -W, & V, & 0, & -R, & Q, \\ W, & 0, & -U, & R, & 0, & -P, \\ -V, & U, & 0, & -Q, & P, & 0, \end{vmatrix},$$

which is obviously equal to zero, showing that there are only five independent relations to be satisfied in our equations, viz: the ratios $u:v:w:p:q:r$.

Reverting now to our value for T, we obtain for U, V, &c., the following values:

10. $$\begin{aligned} U &= a_{11}u + a_{12}v + a_{13}w + a_{14}p + a_{15}q + a_{16}r, \\ V &= a_{21}u + a_{22}v + a_{23}w + a_{24}p + a_{25}q + a_{26}r, \\ W &= a_{31}u + a_{32}v + a_{33}w + a_{34}p + a_{35}q + a_{36}r, \\ P &= a_{41}u + a_{42}v + a_{43}w + a_{44}p + a_{45}q + a_{46}r, \\ Q &= a_{51}u + a_{52}v + a_{53}w + a_{54}p + a_{55}q + a_{56}r, \\ R &= a_{61}u + a_{62}v + a_{63}w + a_{64}p + a_{65}q + a_{66}r. \end{aligned}$$

Denote by ∇ the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix}.$$

According to the well-known conditions that must be satisfied in order that T may be positive, we must have

$$\nabla, \quad \frac{\partial\nabla}{\partial a_{11}}, \quad \frac{\partial^2\nabla}{\partial a_{11}\partial a_{22}} \cdots \frac{\partial^6\nabla}{\partial a_{11}\ldots\partial a_{66}},$$

all positive and different from zero, a point of great importance.

Now from the last equations we can determine the velocities in terms of the forces, and for this determination we have

11.
$$\begin{aligned}
u &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{11}}U + \frac{\partial\nabla}{\partial a_{12}}V + \frac{\partial\nabla}{\partial a_{13}}W + \frac{\partial\nabla}{\partial a_{14}}P + \frac{\partial\nabla}{\partial a_{15}}Q + \frac{\partial\nabla}{\partial a_{16}}R\right], \\
v &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{21}}U + \frac{\partial\nabla}{\partial a_{22}}V + \frac{\partial\nabla}{\partial a_{23}}W + \frac{\partial\nabla}{\partial a_{24}}P + \frac{\partial\nabla}{\partial a_{25}}Q + \frac{\partial\nabla}{\partial a_{26}}R\right], \\
w &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{31}}U + \frac{\partial\nabla}{\partial a_{32}}V + \frac{\partial\nabla}{\partial a_{33}}W + \frac{\partial\nabla}{\partial a_{34}}P + \frac{\partial\nabla}{\partial a_{35}}Q + \frac{\partial\nabla}{\partial a_{36}}R\right], \\
p &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{41}}U + \frac{\partial\nabla}{\partial a_{42}}V + \frac{\partial\nabla}{\partial a_{43}}W + \frac{\partial\nabla}{\partial a_{44}}P + \frac{\partial\nabla}{\partial a_{45}}Q + \frac{\partial\nabla}{\partial a_{46}}R\right], \\
q &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{51}}U + \frac{\partial\nabla}{\partial a_{52}}V + \frac{\partial\nabla}{\partial a_{53}}W + \frac{\partial\nabla}{\partial a_{54}}P + \frac{\partial\nabla}{\partial a_{55}}Q + \frac{\partial\nabla}{\partial a_{56}}R\right], \\
r &= \nabla^{-1}\left[\frac{\partial\nabla}{\partial a_{61}}U + \frac{\partial\nabla}{\partial a_{62}}V + \frac{\partial\nabla}{\partial a_{63}}W + \frac{\partial\nabla}{\partial a_{64}}P + \frac{\partial\nabla}{\partial a_{65}}Q + \frac{\partial\nabla}{\partial a_{66}}R\right].
\end{aligned}$$

For equations 6, it is easy to see that we have the three integrals

12.
$$\begin{aligned}
2T &= L, \\
U^2 + V^2 + W^2 &= M, \\
UP + VQ + WR &= N,
\end{aligned}$$

L, M and N denoting arbitrary constants. These are the general integrals given by Kirchhoff. If we introduce a set of axes ξ, η, ζ fixed in the fluid, we of course have, α, β, γ, $\alpha_1 \ldots \gamma_3$ being functions of the time,

13.
$$\begin{aligned}
\xi &= \alpha + \alpha_1 x + \beta_1 y + \gamma_1 z, \\
\eta &= \beta + \alpha_2 x + \beta_2 y + \gamma_2 z, \\
\zeta &= \gamma + \alpha_3 x + \beta_3 y + \gamma_3 z,
\end{aligned}$$

the quantities u, v, w, p, q, r being connected with the quantities $\alpha, \beta, \ldots$ by the known relations

14.
$$u = \alpha_1 \frac{d\alpha}{dt} + \beta_1 \frac{d\beta}{dt} + \gamma_1 \frac{d\gamma}{dt},$$
&c., &c.
$$p = \alpha_2 \frac{d\alpha_3}{dt} + \beta_2 \frac{d\beta_3}{dt} + \gamma_2 \frac{d\gamma_3}{dt},$$
&c., &c.

and also
$$\frac{d\alpha_1}{dt} = \alpha_3 q - \alpha_2 r, \quad \frac{d\beta_1}{dt} = \beta_3 q - \beta_2 r, \quad \frac{d\gamma_1}{dt} = \gamma_3 q - \gamma_2 r.$$
. . . .

These last nine equations are of the form (as remarked by Clebsch),
$$\frac{dA_1}{dt} = A_3 q - A_2 r,$$
$$\frac{dA_2}{dt} = A_1 r - A_3 p,$$
$$\frac{dA_3}{dt} = A_2 p - A_1 q;$$
from which we have immediately the integral
$$A_1^2 + A_2^2 + A_3^2 = \text{const.},$$
the const. being for our case $= 1$. Multiply now equations 6 by A_1, A_2, A_3 respectively, and add, observing the above relations, and we have
$$\frac{d}{dt}(A_1 U + A_2 V + A_3 W) = 0,$$
which gives us the three Kirchhoffian integrals

15.
$$\alpha_1 U + \alpha_2 V + \alpha_3 W = L',$$
$$\beta_1 U + \beta_2 V + \beta_3 W = M',$$
$$\gamma_1 U + \gamma_2 V + \gamma_3 W = N'.$$

To these integrals for determining the position of the body we can add three more, viz:

16.
$$\alpha_1 P + \alpha_2 Q + \alpha_3 R = l + \beta N' - \gamma M',$$
$$\beta_1 P + \beta_2 Q + \beta_3 R = m + \gamma L' - \alpha N',$$
$$\gamma_1 P + \gamma_2 Q + \gamma_3 R = n + \alpha M' - \beta L',$$

the following relations obviously connecting the constants:

17.
$$L'^2 + M'^2 + N'^2 = M,$$
$$L'l + M'm + N'n = N.$$

The particular case where the body has its mass distributed symmetrically with respect to three mutually perpendicular planes, *i. e.* where T takes the form
$$a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2,$$

has been discussed in a very elegant manner by Weber in a recent number of the Mathematische Annalen. The investigation is made in the first place to depend upon a remarkable property of the ϑ-functions of two variables, and secondly, the author proceeds to the direct integration of the differential equations by means of hyperelliptic integrals.

For the particular case where the body is an ellipsoid, the values of a_{ij}, determined by the formulæ

$$a_{ij} = -\rho\int ds\phi_i \frac{\partial\varphi_j}{\partial n}$$

can be readily seen to depend upon elliptic functions, the integration extending over the entire surface of the ellipsoid of which ds is an element. The case where the body possesses a surface of revolution and a symmetrical distribution of mass, has been very fully discussed by Kirchhoff (in Vol. 71 of Crelle's Journal) who makes the solution of the problem depend upon elliptic functions. The form of T for this case is given by

$$2T = a_{11}u^2 + a_{22}(v^2 + w^2) + a_{44}p^2 + a_{55}(q^2 + r^2),$$

the constants reducing to only four. In Vol. 12 of the Mathematische Annalen, Köpcke has discussed the same problem by aid of the ϑ-functions, obtaining results which are very convenient for numerical computation.

Steady Motion.

The method employed in the following brief examination of the *steady* motion of a solid in a fluid was suggested to me by reading Routh's Essay "On the Stability of a Given State of Motion." I believe the results stated to be new, though I would not venture to make any positive assertion to that effect. I can simply say that the investigation is original and the results obtained seem of interest. "A steady motion is such that the same change of motion follows from the same initial disturbance at whatever instant the disturbance is communicated to the system." The conditions for steady motion of the solid are given by the relations

α. $$\frac{dU}{dt} = \frac{dV}{dt} = \frac{dW}{dt} = 0,$$

β. $$\frac{dP}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0.$$

The conditions α are satisfied by making

18. $$\frac{U}{p} = \frac{V}{q} = \frac{W}{r} = \lambda,$$

or,

18'. $$\frac{a_{11}u + a_{12}v + a_{13}w}{p} = \frac{a_{21}u + a_{22}v + a_{23}w}{q} = \frac{a_{31}u + a_{32}v + a_{33}w}{r} = \lambda .$$

The equations β give now

19. $$\frac{P - \lambda u}{p} = \frac{Q - \lambda v}{q} = \frac{R - \lambda w}{r} = \mu .$$

These last equations can evidently be replaced by

20. $$\frac{UP + VQ + WR - \lambda(uU + vV + wW)}{pU + qV + rW} = \mu .$$

or,

$$(\lambda u + \mu p)\, U + (\lambda v + \mu q)\, V + (\lambda w + \mu r)\, W = \text{const.},$$

which is identical with the known relation

$$UP + VQ + WR = \text{const.}$$

From the expressions for λ and μ we see that it is possible for the body to have an infinite number of steady motions, without making any restrictions as to the form of the body or the distribution of mass in its interior. These steady motions being each produced by a certain *impulse*, will consist in general of a translation in the direction of, and a rotation round, the axis of the component couple. The locus of the system of axes is a ruled surface, whose position with respect to the body is of course invariable. The component velocities of the body with reference to the axes x, y, z fixed in the body are

$$u + zq - yr,$$
$$v + xr - zp,$$
$$w + yp - xq,$$

or, as these may be written

$$\frac{1}{\lambda}\,[P - \mu p + \lambda\,(zq - yr)],$$
$$\frac{1}{\lambda}\,[Q - \mu q + \lambda\,(xr - zp)],$$
$$\frac{1}{\lambda}\,[R - \mu r + \lambda\,(yp - xq)].$$

Substituting for U, V, &c., their values in the equations giving λ and μ, we obtain the system of linear equations

21. $$\begin{gathered} a_{11}u + a_{12}v + a_{13}w + (a_{14} - \mu)\, p + a_{15}q + a_{16}r = 0, \\ a_{21}u + a_{22}v + a_{23}w + a_{24}p + (a_{25} - \lambda)\, q + a_{26}r = 0, \\ a_{31}u + a_{32}v + a_{33}w + a_{34}p + a_{35}q + (a_{36} - \lambda)\, r = 0, \\ (a_{41} - \lambda)\, u + a_{42}v + a_{43}w + (a_{44} - \mu)\, p + a_{45}q + a_{46}r = 0, \\ a_{51}u + (a_{52} - \lambda)\, v + r_{53}w + a_{54}p + (a_{55} - \mu)\, q + a_{56}r = 0, \\ a_{61}u + a_{62}v + (a_{63} - \lambda)\, w + a_{64}p + a_{65}q + (a_{66} - \mu)\, r = 0. \end{gathered}$$

Eliminating the ratios $u:v:$ &c., we have for the relation connecting the quantities λ and μ

$$22.\qquad \nabla_{\lambda\mu} \equiv \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}-\lambda, & a_{15}, & a_{16} \\ a_{21}, & a_{22}, & a_{23}, & a_{24}, & a_{25}-\lambda, & a_{26} \\ a_{31}, & a_{32}, & a_{33}, & a_{34}, & a_{35}, & a_{36}-\lambda \\ a_{41}-\lambda, & a_{42}, & a_{43}, & a_{44}-\mu, & a_{45}, & a_{46} \\ a_{51}, & a_{52}-\lambda, & a_{53}, & a_{54}, & a_{55}-\mu, & a_{56} \\ a_{61}, & a_{62}, & a_{63}-\lambda, & a_{64}, & a_{65}, & a_{66}-\mu \end{vmatrix} = 0.$$

This equation affords us the means of determining either λ or μ, provided we assume a determinate value for one of these quantities. Assume for λ some arbitrary value, then the equation $\nabla_{\lambda\mu} = 0$ is of the third degree in μ, and we have thus, for any one value of λ, three corresponding values of μ. We have now, for the complete specification of the motion,

$$23.\qquad u:v:w:p:q:r =$$

$$\frac{\partial \nabla_{\lambda\mu}}{\partial a_{11}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{12}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{13}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{14}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{15}} : \frac{\partial \nabla_{\lambda\mu}}{\partial a_{16}},$$

substituting in the minors $\frac{\partial \nabla_{\lambda\mu}}{\partial a_{16}}$ the assumed value of λ and the determined value of μ.

If we give μ a determinate value, we have $\nabla_{\lambda\mu} = 0$, an equation of the sixth degree for finding the corresponding values of λ. If we make $\mu = 0$, it is known that the roots of $\nabla_{\lambda\mu} = 0$ will all be real, three positive and three negative, and the motion in this case will be completely determined.

For simplicity we may assume the axes of x, y, z parallel to the three directions of permanent translation, which is equivalent to making

$$a_{12} = a_{13} = a_{23} = 0;$$

we have then from the first three of equations 16,

$$24.\qquad \begin{aligned} u &= \frac{(a_{14}-\lambda)p + a_{15}q + a_{16}r}{a_{11}}, \\ v &= \frac{a_{24}p + (a_{25}-\lambda)q + a_{26}r}{a_{22}}, \\ w &= \frac{a_{34}p + a_{35}q + (a_{36}-\lambda)r}{a_{33}}, \end{aligned}$$

Substituting these values in the last three of equations 16, we have

$$\left[\frac{(a_{14}-\lambda)^2}{a_{11}}+\frac{a_{24}^2}{a_{22}}+\frac{a_{34}^2}{a_{33}}+a_{44}\right]p+\left[\frac{(a_{14}-\lambda)\,a_{15}}{a_{11}}+\frac{a_{24}(a_{25}-\lambda)}{a_{22}}+\frac{a_{34}a_{35}}{a_{33}}+a_{45}\right]q$$
$$+\left[\frac{(a_{14}-\lambda)\,a_{16}}{a_{11}}+\frac{a_{24}a_{26}}{a_{22}}+\frac{a_{34}(a_{36}-\lambda)}{a_{33}}+a_{46}\right]r=\mu p,$$
$$\left[\frac{(a_{14}-\lambda)\,a_{15}}{a_{11}}+\frac{a_{24}(a_{25}-\lambda)}{a_{22}}+\frac{a_{34}a_{35}}{a_{33}}+a_{45}\right]p+\left[\frac{a_{15}^2}{a_{11}}+\frac{(a_{25}-\lambda)^2}{a_{22}}+\frac{a_{35}^2}{a_{33}}+a_{55}\right]q$$
$$+\left[\frac{a_{15}a_{16}}{a_{11}}+\frac{(a_{25}-\lambda)\,a_{26}}{a_{22}}+\frac{a_{35}(a_{36}-\lambda)}{a_{33}}+a_{56}\right]r=\mu q,$$
$$\left[\frac{(a_{14}-\lambda)\,a_{16}}{a_{11}}+\frac{a_{24}a_{26}}{a_{22}}+\frac{a_{34}(a_{36}-\lambda)}{a_{33}}+a_{46}\right]p+\left[\frac{a_{15}a_{16}}{a_{11}}+\frac{(a_{25}-\lambda)\,a_{26}}{a_{22}}+\frac{a_{35}(a_{36}-\lambda)}{a_{33}}+a_{56}\right]q$$
$$+\left[\frac{a_{16}^2}{a_{11}}+\frac{a_{26}^2}{a_{22}}+\frac{(a_{36}-\lambda)^2}{a_{33}}+a_{66}\right]r=\mu r.$$

These may obviously be written briefly in the form

25. $$\begin{aligned}Ep+G'q+F'r&=\mu p\,,\\ G'p+Fq+E'r&=\mu q\,,\\ F'p+E'q+Gr&=\mu r\,.\end{aligned}$$

We have then for μ the cubic

26. $$\begin{vmatrix} E-\mu, & G', & F' \\ G', & F-\mu, & E' \\ F', & E', & G-\mu \end{vmatrix}=0\,.$$

Thus the directions of the three axes corresponding to an assumed value of λ are at right angles to each other, but need not intersect; and, in general, no two values of μ will coincide with each other. Taking then a series of values of λ, and finding the corresponding values of μ, we will have, as the locus of the axes, a ruled surface of three distinct nappes.

Assume, in equations 6, that the only force which acts upon the body is the couple whose components are P, Q and R, *i. e.* make

$$U=V=W=0\,;$$

the equations are then satisfied by writing

27. $$\frac{P}{p}=\frac{Q}{q}=\frac{R}{r}=\mu\,,$$

each of these ratios being the value of μ for $\lambda=0$. Calling μ_1, μ_2, μ_3 the three corresponding values of μ, write

28. $$\begin{aligned}P&=\mu p\,,\\ Q&=\mu q\,,\\ R&=\mu r\,.\end{aligned}$$

These, substituted in equations 6, give us

$$\mu_1 \frac{dp}{dt} = (\mu_2 - \mu_3)\, qr\,,$$

29. $$\mu_2 \frac{dq}{dt} = (\mu_3 - \mu_1)\, rp\,,$$

$$\mu_3 \frac{dr}{dt} = (\mu_1 - \mu_2)\, pq\,;$$

these three equations are identical in form with Euler's equations for the rotation of a rigid body, and their solution can, consequently, be regarded as known. The coefficients μ_1, μ_2, μ_3 depend, in our case, not only upon the body but upon the density of the fluid;—if this latter was supposed equal to zero, the problem would coincide with that of the rotation of a free rigid body.

Assume that the body possesses three planes of symmetry, or, to fix the idea, assume the body to be an ellipsoid. The expression for the energy becomes in this case

$$2T = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2\,,$$

the other terms all vanishing since the sign of T must remain unchanged if we change u into $-u$, v into $-v$, &c. We have now

$$U = a_{11}u\,, \quad P = a_{44}p\,,$$

30. $$V = a_{22}v\,, \quad Q = a_{55}q\,,$$

$$W = a_{33}w\,, \quad R = a_{66}r\,,$$

giving, of course, for steady motion,

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0\,,$$

$$\frac{dp}{dt} = \frac{dq}{dt} = \frac{dr}{dt} = 0\,,$$

or the axes of steady motion are the axes of the ellipsoid.

Motion of the Fluid.

The equations giving the motions of the fluid particles relatively to the body are

$$\frac{dx}{dt} = \frac{\partial\varphi}{\partial x} - u - zq + yr\,,$$

31. $$\frac{dy}{dt} = \frac{\partial\varphi}{\partial y} - v - xr + zp\,,$$

$$\frac{dz}{dt} = \frac{\partial\varphi}{\partial z} - w - yp + xq\,,$$

ϕ denoting the velocity potential.

In the investigation of the motion of the fluid particles due to the motion of a body of given form, it is often desirable to transform the equations of motion from rectangular to curvilinear coordinates, and to this transformation we will now turn our attention. Taking the quantities $\lambda_1, \lambda_2, \lambda_3$ as the variable parameters in a certain system of curvilinear coordinates, let us suppose that we have the relations

32. $$\lambda_1 = F_1(x, y, z), \quad \lambda_2 = F_2(x, y, z), \quad \lambda_3 = F_3(x, y, z),$$ and 33. $$x = f_1(\lambda_1, \lambda_2, \lambda_3), \quad y = f_2(\lambda_1, \lambda_2, \lambda_3), \quad z = f_3(\lambda_1, \lambda_2, \lambda_3).$$

The known conditions that the surfaces $\lambda_1 = \text{const.}$, $\lambda_2 = \text{const.}$, $\lambda_3 = \text{const.}$, should be orthogonal are

34. $$\frac{\partial\lambda_1}{\partial x}\frac{\partial\lambda_2}{\partial x} + \frac{\partial\lambda_1}{\partial y}\frac{\partial\lambda_2}{\partial y} + \frac{\partial\lambda_1}{\partial z}\frac{\partial\lambda_2}{\partial z} = 0, \text{ \&c.,}$$

and

35. $$\frac{\partial x}{\partial\lambda_1}\frac{\partial x}{\partial\lambda_2} + \frac{\partial y}{\partial\lambda_1}\frac{\partial y}{\partial\lambda_2} + \frac{\partial z}{\partial\lambda_1}\frac{\partial z}{\partial\lambda_2} = 0, \text{ \&c.,}$$

We have further,

36. $$\begin{aligned} dx &= \frac{\partial f_1}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_1}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_1}{\partial\lambda_3} d\lambda_3, \\ dy &= \frac{\partial f_2}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_2}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_2}{\partial\lambda_3} d\lambda_3, \\ dz &= \frac{\partial f_3}{\partial\lambda_1} d\lambda_1 + \frac{\partial f_3}{\partial\lambda_2} d\lambda_2 + \frac{\partial f_3}{\partial\lambda_3} d\lambda_3. \end{aligned}$$

These give, as is well known,

$$\begin{aligned} E^2 d\lambda_1 &= \frac{\partial x}{\partial\lambda_1} dx + \frac{\partial y}{\partial\lambda_1} dy + \frac{\partial z}{\partial\lambda_1} dz, \\ F^2 d\lambda_2 &= \frac{\partial x}{\partial\lambda_2} dx + \frac{\partial y}{\partial\lambda_2} dy + \frac{\partial z}{\partial\lambda_2} dz, \\ G^2 d\lambda_3 &= \frac{\partial x}{\partial\lambda_3} dx + \frac{\partial y}{\partial\lambda_3} dy + \frac{\partial z}{\partial\lambda_3} dz, \end{aligned}$$

whence

37. $$\begin{aligned} E^2 &= \left(\frac{\partial x}{\partial\lambda_1}\right)^2 + \left(\frac{\partial y}{\partial\lambda_1}\right)^2 + \left(\frac{\partial z}{\partial\lambda_1}\right)^2, \\ F^2 &= \left(\frac{\partial x}{\partial\lambda_2}\right)^2 + \left(\frac{\partial y}{\partial\lambda_2}\right)^2 + \left(\frac{\partial z}{\partial\lambda_2}\right)^2, \\ G^2 &= \left(\frac{\partial x}{\partial\lambda_3}\right)^2 + \left(\frac{\partial y}{\partial\lambda_3}\right)^2 + \left(\frac{\partial z}{\partial\lambda_3}\right)^2, \end{aligned}$$

and

38.

$$
\begin{aligned}
\frac{1}{E^2} &= \left(\frac{\partial\lambda_1}{\partial x}\right)^2 + \left(\frac{\partial\lambda_1}{\partial y}\right)^2 + \left(\frac{\partial\lambda_1}{\partial z}\right)^2, \\
\frac{1}{F^2} &= \left(\frac{\partial\lambda_2}{\partial x}\right)^2 + \left(\frac{\partial\lambda_2}{\partial y}\right)^2 + \left(\frac{\partial\lambda_2}{\partial z}\right)^2, \\
\frac{1}{G^2} &= \left(\frac{\partial\lambda_3}{\partial x}\right)^2 + \left(\frac{\partial\lambda_3}{\partial y}\right)^2 + \left(\frac{\partial\lambda_3}{\partial z}\right)^2.
\end{aligned}
$$

The F entering here has, of course, no connection with the F in the above functional expressions. Our equations for the motion of a fluid particle now give, by very easy reductions,

39.

$$
\begin{aligned}
E^2\frac{d\lambda_1}{dt} &= \frac{\partial\varphi}{\partial\lambda_1} - \left(u\frac{\partial x}{\partial\lambda_1} + v\frac{\partial y}{\partial\lambda_1} + w\frac{\partial z}{\partial\lambda_1}\right) - \left(py\frac{\partial z}{\partial\lambda_1}\right), \\
F^2\frac{d\lambda_2}{dt} &= \frac{\partial\varphi}{\partial\lambda_2} - \left(u\frac{\partial x}{\partial\lambda_2} + v\frac{\partial y}{\partial\lambda_2} + w\frac{\partial z}{\partial\lambda_2}\right) - \left(py\frac{\partial z}{\partial\lambda_2}\right), \\
G^2\frac{d\lambda_3}{dt} &= \frac{\partial\varphi}{\partial\lambda_3} - \left(u\frac{\partial x}{\partial\lambda_3} + v\frac{\partial y}{\partial\lambda_3} + w\frac{\partial z}{\partial\lambda_3}\right) - \left(py\frac{\partial z}{\partial\lambda_3}\right),
\end{aligned}
$$

where

40.

$$
\left(py\frac{\partial z}{\partial\lambda_i}\right) = \begin{vmatrix} p, & q, & r \\ x, & y, & z \\ \frac{\partial x}{\partial\lambda_i}, & \frac{\partial y}{\partial\lambda_i}, & \frac{\partial z}{\partial\lambda_i} \end{vmatrix}.
$$

Suppose we have now a system of orthogonal surfaces of the second order given by the equations

41.

$$
\begin{aligned}
\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} + \frac{z^2}{c^2+\lambda_1} &= 1, \\
\frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} + \frac{z^2}{c^2+\lambda_2} &= 1, \\
\frac{x^2}{a^2+\lambda_3} + \frac{y^2}{b^2+\lambda_3} + \frac{z^2}{c^2+\lambda_3} &= 1,
\end{aligned}
$$

where, for $\lambda_1 = 0$ the first equation becomes the equation of the surface of the body if this is an ellipsoid.

The values of x, y and z, derivable from these equations, are, of course, well known, but the following direct transformation, which I believe to be new, is so brief, and depends upon such elementary properties of a system of linear equations, that I give it before proceeding further with the problem in hand. Consider the group of n linear equations:

42.

$$
\begin{aligned}
&\frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} + \frac{x_3}{\alpha_3} + \ldots \frac{x_n}{\alpha_n} = 1, \\
&\frac{x_1}{\beta_1} + \frac{x_2}{\beta_2} + \frac{x_3}{\beta_3} + \ldots \frac{x_n}{\beta_n} = 1, \\
&\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\
&\frac{x_1}{\nu_1} + \frac{x_2}{\nu_2} + \frac{x_3}{\nu_3} + \ldots \frac{x_n}{\nu_n} = 1,
\end{aligned}
$$

having the following relations among the quantities α, β, &c.:

$$\alpha_1 - \alpha_2 = \beta_1 - \beta_2 = \gamma_1 - \gamma_2 = \ldots \nu_1 - \nu_2\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

43. $$\alpha_1 - \alpha_i = \beta_1 - \beta_i = \gamma_1 - \gamma_i = \ldots \nu_1 - \nu_i\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_1 - \alpha_n = \beta_1 - \beta_n = \gamma_1 - \gamma_n = \ldots \nu_1 - \nu_n\,,$$

and

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \alpha_3 - \beta_3 = \ldots \alpha_n - \beta_n\,,$$

&c., &c., &c.

Designate by ∇ the eliminant of our set of equations—that is,

$$\nabla = \begin{vmatrix} \alpha_1^{-1}, & \alpha_2^{-1}, & \alpha_3^{-1}, \ldots & \alpha_n^{-1} \\ \beta_1^{-1}, & \beta_2^{-1}, & \beta_3^{-1}, \ldots & \beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ \nu_1^{-1}, & \nu_2^{-1}, & \nu_3^{-1}, \ldots & \nu_n^{-1} \end{vmatrix}.$$

Then we have for the values of x_1, x_2, &c.,

44. $$\begin{aligned} x_1 &= \nabla^{-1}\left[\frac{\partial\Gamma}{\partial\alpha_1^{-1}} + \frac{\partial\Gamma}{\partial\beta_1^{-1}} + \ldots \frac{\partial\Gamma}{\partial\nu_1^{-1}}\right], \\ x_2 &= \nabla^{-1}\left[\frac{\partial\Gamma}{\partial\alpha_2^{-1}} + \frac{\partial\Gamma}{\partial\beta_2^{-1}} + \ldots \frac{\partial\Gamma}{\partial\nu_2^{-1}}\right], \\ &\cdots\cdots\cdots\cdots \\ \alpha_i &= \nabla^{-1}\left[\frac{\partial\Gamma}{\partial\alpha_i^{-1}} + \frac{\partial\Gamma}{\partial\beta_i^{-1}} + \ldots \frac{\partial\Gamma}{\partial\nu_i^{-1}}\right], \end{aligned}$$

&c., &c.

For the further determination of these quantities, write the determinant in the form

$$\nabla = \alpha_1^{-1}\beta_1^{-1}\gamma_1^{-1}\ldots\nu_1^{-1}\begin{vmatrix} 1, & \alpha_1\alpha_2^{-1}, & \alpha_1\alpha_3^{-1}, \ldots & \alpha_1\alpha_n^{-1} \\ 1, & \beta_1\beta_2^{-1}, & \beta_1\beta_3^{-1}, \ldots & \beta_1\beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \nu_1\nu_2^{-1}, & \nu_1\nu_3^{-1}, \ldots & \nu_1\nu_n^{-1} \end{vmatrix},$$

or again, by subtracting the first column from each of the others,

$$\nabla = \alpha_1^{-1}\beta_1^{-1}\gamma_1^{-1}\ldots\nu_1^{-1}\begin{vmatrix} 1, & \dfrac{\alpha_1-\alpha_2}{\alpha_2}, & \dfrac{\alpha_1-\alpha_3}{\alpha_3}, \ldots & \dfrac{\alpha_1-\alpha_n}{\alpha_n} \\ 1, & \dfrac{\beta_1-\beta_2}{\beta_2}, & \dfrac{\beta_1-\beta_3}{\beta_3}, \ldots & \dfrac{\beta_2-\beta_n}{\beta_n} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \dfrac{\nu_1-\nu_2}{\nu_2}, & \dfrac{\nu_1-\nu_3}{\nu_3}, \ldots & \dfrac{\nu_1-\nu_n}{\nu_n} \end{vmatrix}.$$

This becomes, by virtue of the assumed relations existing between the differences of the quantities, $\alpha, \beta \ldots \nu$,

$$\nabla = \frac{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}{a_1\beta_1\gamma_1\ldots\nu_1}\begin{vmatrix} 1, & \alpha_2^{-1}, & \alpha_3^{-1}, & \ldots \alpha_n^{-1} \\ 1, & \beta_2^{-1}, & \beta_3^{-1}, & \ldots \beta_n^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \nu_2^{-1}, & \nu_3^{-1}, & \ldots \nu_n^{-1} \end{vmatrix}.$$

The principal minors of this determinant, with reference to the elements in the first column, are the same as the corresponding principal minors of our original determinant; therefore, we have

45. $$\nabla = \frac{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}{a_1\beta_1\gamma_1\ldots\nu_1}\left[\frac{\partial\Gamma}{\partial a_1^{-1}} + \frac{\partial\Gamma}{\partial\beta_1^{-1}} + \ldots \frac{\partial\Gamma}{\partial\nu_1^{-1}}\right],$$

from which follows

46. $$x_1 = \frac{a_1\beta_1\gamma_1\ldots\nu_1}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)},$$

and in general

$$x_i = \frac{a_i\beta_i\gamma_i\ldots\nu_i}{(a_i-a_1)(a_i-a_2)\ldots(a_i-a_n)}.$$

These are of the same form as the expressions obtained by Jacobi and given on page 202 of the "*Vorlesungen über Dynamik.*" Let σ and τ denote any two of the constants $\alpha, \beta, \gamma \ldots \nu$; then we have, by subtracting the equation whose coefficients are $\frac{1}{\sigma}$ from that having the coefficients $\frac{1}{\tau}$,

$$\frac{x_1(\sigma_1-\tau_1)}{\tau_1\sigma_1} + \frac{x_2(\sigma_2-\tau_2)}{\tau_2\sigma_2} + \ldots \frac{x_n(\sigma_n-\tau_n)}{\tau_n\sigma_n} = 0,$$

or, since $\tau_1 - \sigma_1 = \sigma_2 - \tau_2 = \&c.$,

$$\frac{x_1}{\tau_1\sigma_1} + \frac{x_2}{\tau_2\sigma_2} + \ldots \frac{x_n}{\tau_n\sigma_n} = 0,$$

which is the same as equation (4) of the "*Vorlesungen.*" We have now the series of equations

47. $$\begin{aligned} &\frac{x_1}{a_1\beta_1} + \frac{x_2}{a_2\beta_2} + \ldots \frac{x_n}{a_n\beta_n} = 0, \\ &\frac{x_1}{a_1\gamma_1} + \frac{x_2}{a_2\gamma_2} + \ldots \frac{x_n}{a_n\gamma_n} = 0, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\frac{x_1}{a_1\nu_1} + \frac{x_2}{a_2\nu_2} + \ldots \frac{x_n}{a_n\nu_n} = 0. \end{aligned}$$

Define the quantity p_τ by the equation

$$p_\tau = \frac{x_1}{\tau_1^2} + \frac{x_2}{\tau_2^2} + \frac{x_3}{\tau_3^2} + \ldots \frac{x_n}{\tau_n^2},$$

and in particular

$$p_\alpha = \frac{x_1}{a_1^2} + \frac{x_2}{a_2^2} + \ldots \frac{x_n}{a_n^2}.$$

Multiply this last by $\frac{\partial \Gamma}{\partial a_1^{-1}}$, and the equations 47 by $\frac{\partial \Gamma}{\partial \beta_1^{-1}}$, $\frac{\partial \Gamma}{\partial \gamma_1^{-1}}$, &c., respectively, and add the products; we have then

$$\frac{\partial \Gamma}{\partial a_1^{-1}} p_\alpha = \frac{x_1}{a_1}\left[\frac{1}{a_1}\frac{\partial \Gamma}{\partial a_1^{-1}} + \frac{1}{\beta_1}\frac{\partial \Gamma}{\partial \beta_1^{-1}} + \ldots \frac{1}{\nu_1}\frac{\partial \Gamma}{\partial \nu_1^{-1}}\right]$$

all the other terms vanishing by virtue of the well-known properties of determinants, that is

$$p_\alpha \frac{\partial \Gamma}{\partial a_1^{-1}} = \frac{x_1}{a_1}\nabla,$$

and, similarly, we obtain

$$p_\alpha \frac{\partial \Gamma}{\partial a^{-1}} = \frac{x_2}{a_2}\nabla,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$p_\alpha \frac{\partial \Gamma}{\partial a_n^{-1}} = \frac{x_n}{a_n}\nabla.$$

Adding these, we have, since

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \ldots \frac{x_n}{a_n} = 1,$$

$$p_\alpha = \frac{\Gamma}{\frac{\partial \Gamma}{\partial a_1^{-1}} + \frac{\partial \Gamma}{\partial a_2^{-1}} + \ldots \frac{\partial \Gamma}{\partial a_n^{-1}}},$$

Now, by merely changing rows into columns, and conversely, in the determinant ∇, we readily see that we must have, by virtue of equation 45,

$$\nabla = \frac{(a_1 - \beta_1)(a_1 - \gamma_1)\ldots(a_1 - \nu_1)}{a_1 a_2 a_3 \ldots a_n}\left[\frac{\partial \Gamma}{\partial a_1^{-1}} + \frac{\partial \Gamma}{\partial a_2^{-1}} + \ldots \frac{\partial \Gamma}{\partial a_n^{-1}}\right];$$

this reduces the above value of p_α to

48. $$p_\alpha = \frac{(a_1 - \beta_1)(a_1 - \gamma_1)\ldots(a_1 - \nu_1)}{a_1 a_2 a_3 \ldots a_n},$$

and gives, for the general value p_τ,

49. $$p_\tau = \frac{(\tau_1 - a_1)(\tau_1 - \beta_1)\ldots(\tau_1 - \nu_1)}{\tau_1 \tau_2 \tau_3 \ldots \tau_n}.$$

All of the other relations existing between the quantities x_1, x_2, &c., and a, β, ... ν can be readily obtained, but it would be out of place to continue

the investigation any further in this paper. The values of x^2, y^2 and z^2 obtained from equation 41, by application of the formulæ of 46, are now

$$x^2 = \frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = \frac{(b^2+\lambda_1)(b^2+\lambda_2)(b^2+\lambda_3)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2 = \frac{(c^2+\lambda_1)(c^2+\lambda_2)(c^2+\lambda_3)}{(c^2-a^2)(c^2-b^2)};$$

and also for E, F and G we have

$$E^2 = \frac{1}{4}\cdot\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)},$$

$$F^2 = \frac{1}{4}\cdot\frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)},$$

$$G^2 = \frac{1}{4}\cdot\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)},$$

The equation of continuity $\nabla^2\phi = 0$ also takes the well-known form

$$(\lambda_2-\lambda_3)\frac{\partial^2\varphi}{\partial\omega_1^2} + (\lambda_3-\lambda_1)\frac{\partial^2\varphi}{\partial\omega_2^2} + (\lambda_1-\lambda_2)\frac{\partial^2\varphi}{\partial\omega_3^2} = 0,$$

where

$$\omega_1 = \int\frac{\partial\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$

ω_2 and ω_3 containing λ_2 and λ_3, respectively, instead of λ_1. We are now in the position to examine the case of the motion of an ellipsoid in the fluid, and the resulting motion of the fluid particles, subjects which will be treated fully in a subsequent paper. The transformed equations of motion, as I have given them here, differ in form from those given by Clebsch, and have been obtained by a slightly different process, but will of course lead to the same results as do those of Clebsch. The investigation of the form of ϕ for this case is given in a very elegant manner by Kirchhoff in his Physik.

BALTIMORE, *March 28th, 1879.*

Sur l'Analyse indéterminée du troisième degré.—Demonstration de plusieurs théorèmes de M. Sylvester.

PAR EDOUARD LUCAS.

SECTION 1.

L'ARITHMÉTIQUE DE DIOPHANTE renferme le premier exemple connu d'Analyse indéterminée du troisème degré; l'immortel auteur y pose, en effet, le problème de trouver deux nombres entiers ou fractionnaires, dont la somme ou la différence de leurs cubes soit égale à la somme ou à la différence des cubes de deux nombres donnés.

FERMAT a indiqué, le premier, un procédé qui permet de déduire, d'une solution initiale, une série indéfinie de solutions nouvelles. Pour résoudre, en nombres entiers ou fractionnaires, l'équation

$$x^3 + y^3 = a^3 + b^3,$$

dans laquelle a et b sont donnés, il suffit de poser

$$x = a + zu, \quad y = b + u,$$

et de disposer de z, de manière à faire disparaître, après la substitution, la première puissance de u. On trouve alors une relation de la forme

$$Au^3 + Bu^2 = 0,$$

qui permet de déterminer u par une équation du premier degré; FERMAT calcule ainsi x et y, et fait servir ces valeurs à la recherche de nouvelles solutions, en nombre indéfini.

Nous remplacerons, dans ce qui suit, les inconnues rationnelles, par des inconnues entières. Désignons par (x, y, z) une première solution, en nombres, entiers, de l'équation

$$x^3 + y^3 = Az^3, \tag{1}$$

nous obtiendrons une autre solution, par le procédé indiqué plus haut, au moyen des formules

$$X = x(x^3 + 2y^3), \quad Y = -y(y^3 + 2x^3), \quad Z = z(x^3 - y^3). \tag{2}$$

On trouve ainsi, successivement, pour $A = 9$,

$$\begin{cases} x_1 = 2, & x_2 = 20, & x_3 = 1\,88479, & x_4 = 12\,43617\,73399\,00948\,36481, \\ y_1 = 1, & y_2 = -17, & y_3 = -36520, & y_4 = 4\,87267\,17171\,43523\,36560, \\ z_1 = 1; & z_2 = 7; & z_3 = +90391; & z_4 = 6\,09623\,83567\,61372\,97449; \end{cases}$$

et, pour $A = 28$,

$$\begin{cases} x_1 = 3, & x_2 = 87, & x_3 = 632\,84705, & x_4 = 18\,92071\,22047\,02010\,97176\,90323\,50335, \\ y_1 = 1, & y_2 = -55, & y_3 = 283\,40511, & y_4 = -15\,01104\,22682\,05492\,03687\,05693\,29391, \\ z_1 = 1; & z_2 = 26; & z_3 = 214\,46828; & z_4 = 4\,94756\,15518\,27392\,93262\,16777\,53432. \end{cases}$$

On observera que ces solutions croissent très-rapidement, et contiennent à peu près quatre fois plus de chiffres, que la solution précédente.

On peut encore remplacer les formules (2) par les suivantes, qui n' en diffèrent que par la forme. Désignons par (x, y, z) des nombres entiers qui vérifient l'équation

$$x^3 + y^3 = Az^3,$$

nous obtiendrons des nombres entiers (X, Y, Z), tels que l'on ait

$$\frac{X^3 + Y^3}{Z^3} = \frac{x^3 + y^3}{z^3} = A,$$

par les formules

$$\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = 0, \quad Xx^2 + Yy^2 = AZz^2.$$

SECTION 2.

LAGRANGE et CAUCHY ont étendu la méthode que nous venons d'indiquer, à des équations du troisième degré beaucoup plus générales. Soit l'équation

$$Ax^3 + By^3 + Cz^3 + 3Dxyz = 0; \tag{3}$$

on déduit d'une première solution (x, y, z), en nombres entiers, une autre solution (X, Y, Z), par les formules

$$\begin{cases} X = x\,(By^3 - Cz^3), \\ Y = y\,(Cz^3 - Ax^3), \\ Z = z\,(Ax^3 - By^3), \end{cases} \tag{4}$$

Ainsi l'équation

$$x^3 + 2y^3 + 3z^3 = 6xyz,$$

qui a pour solution immédiate

$$x_0 = y_0 = z_0 = 1,$$

donne ensuite les solutions

$$\begin{cases} x_1 = 1, & x_2 = 19, & x_3 = 2\,82473, \ldots \\ y_1 = -2, & y_2 = 4, & y_3 = -86392, \ldots \\ z_1 = 1; & z_2 = -17; & z_3 = -1\,14427; \ldots \end{cases}$$

Nous observerons que les formules (4) peuvent être remplacées par celles-ci

$$(5)\qquad \frac{X}{x}+\frac{Y}{y}+\frac{Z}{z}=0,\quad AXx^2+BYy^2+CZz^2=0,$$

et conduisent à l'identité

$$Ax^3(Ax^3+2By^3)^3+By^3(By^3+2Ax^3)^3+27A^2B^2x^6y^6=[A^2x^6+7ABx^3y^3+B^2y^6]^2.$$

Cette identité fournit ainsi une série indéfinie de solutions de l'équation indéterminée

$$Au^3+Bv^3+A^2B^2w^3=t^2.$$

On doit encore à CAUCHY, l'indication suivante :* Si (x_0, y_0, z_0) et (x_1, y_1, z_1) désignent deux solutions distinctes de l'équation (3), on obtient une solution nouvelle au moyen des formules

$$\left\{\begin{aligned} X &= By_0y_1\,(x_0y_1-x_1y_0)+Cz_0z_1\,(x_0z_1-z_0x_1)+D\,(x_0^2y_1z_1-x_1^2y_0z_0),\\ Y &= Cz_0z_1\,(y_0z_1-y_1z_0)+Ax_0x_1\,(y_0x_1-x_0y_1)+D\,(y_0^2z_1x_1-y_1^2z_0x_0),\\ Z &= Ax_0x_1\,(z_0x_1-z_1x_0)+By_0y_1\,(z_0y_1-y_0z_1)+D\,(z_0^2x_1y_1-z_1^2x_0y_0).\end{aligned}\right.$$

On peut remplacer ces formules par celles-ci :

$$(6)\qquad \begin{vmatrix} X, & Y, & Z, \\ x_0, & y_0, & z_0, \\ x_1, & y_1, & z_1, \end{vmatrix}=0,\quad AXx_0x_1+BYy_0y_1+CZz_0z_1=0.$$

Ainsi, par exemple, les solutions (x_0, y_0, z_0) et (x_2, y_2, z_2) de l'équation numérique, que nous venons de considérer, donnent

$$X=143,\quad Y=113,\quad Z=71.$$

SECTION 3.

Les résultats précédents sont des cas particuliers de ceux que nous allons indiquer. Soit l'équation du troisième degré

$$(7)\qquad f(x, y, z)=0,$$

d'une courbe en coordonnées rectilignes et homogènes ; désignons par m_1 un point dont les coordonnées (x_1, y_1, z_1) sont rationnelles, et qu'il est facile de rendre entières ; on a ainsi une première solution, en nombres entiers, de l'équation proposée. On obtient de nouvelles solutions, par l'un des trois procédés suivants :

1°. Si l'on mène la tangente à la cubique en m_1, cette droite rencontre la courbe en un autre point m dont les coordonnées sont rationnelles ; par

*CAUCHY.—*Sur la résolution de quelques équations indéterminées—en nombres entiers.*—Exercices de Mathématiques, 1826, t. I, pag. 256.

conséquent, d'une première solution de l'équation (7) on déduit, en général, une autre solution, par les formules

$$f(x, y, z) = 0, \quad x\frac{df}{dx_1} + y\frac{df}{dy_1} + z\frac{df}{dz_1} = 0.$$

Cependant, lorsque la tangente est parallèle à l'une des asymptotes de la cubique, ou lorsque la tangente est menée par un point d'inflexion, on n'obtient pas de solutions nouvelles.

2°. Si m_1 et m_2 désignent deux points de la cubique dont les coordonnées (x_1, y_1, z_1) et (x_2, y_2, z_2) sont entières, on obtient, en général, une nouvelle solution de l'équation (7), en prenant l'intersection de la courbe avec la sécante m_1m_2; on a donc à résoudre les deux équations

$$f(x, y, z) = 0, \quad \begin{vmatrix} x, & y, & z, \\ x_1, & y_1, & z_1, \\ x_2, & y_2, & z_2, \end{vmatrix} = 0,$$

en tenant compte des relations

$$f(x_1; y_1, z_1) = 0, \quad f(x_2, y_2, z_2) = 0.$$

3°. Lorsque l'on connaît cinq solutions de l'équation (7), on obtient, en général, une sixième solution, en prenant le point d'intersection avec la cubique, de la conique passant par les cinq points qui correspondent aux solutions données. D'ailleurs, on peut supposer plusieurs de ces points réunis en un seul, et en particulier tous les cinq réunis en un seul, à la condition d'établir entre les deux courbes le contact correspondant.

Nous observerons que les méthodes de FERMAT, LAGRANGE, et CAUCHY reviennent aux deux premiers procédés.

SECTION 4.

Nous considérerons, plus particulièrement, dans ce qui suit l'équation (1). EULER et LEGENDRE ont démontré que l'équation

$$(1) \qquad x^3 + y^3 = Az^3,$$

est impossible, lorsque A est égal à 1, 2, 3, 4 ou 5; mais LEGENDRE s'est trompé, pour le cas de $A = 6$, ainsi que nous le montrerons plus loin. M. SYLVESTER est venu ajouter une importante contribution à la théorie de cette équation, en donnant un certain nombre de formes générales de A pour lesquelle, l'équation (1) est impossible. Les divers théorèmes indiqués par M. SYLVESTER sont renfermés dans l'énoncé suivant:

Si p et q désignent des nombres premiers des formes respectives $18n+5$ et $18n+11$, *il est impossible de décomposer en deux cubes, soit entiers, soit fractionnaires, aucun des nombres A suivants:*

$$p,\ 2p,\ 4p^2;\quad q^2,\ 2q^2,\ 4q.$$

PREMIER CAS.—En effet, soit d'abord à résoudre l'équation indéterminée

$$x^3+y^3=Az^3, \tag{1}$$

dans laquelle A désigne un nombre premier p de la forme $18n+5$, ou le carré q^2 d'un nombre premier de la forme $18n+11$; nous pouvons supposer les entiers x, y, z, premiers entre eux. Mais le cube d'un nombre entier divisé par 9 donne pour reste 0, ou $+1$ ou -1; donc, pour que l'équation (1) soit possible, il faut que z^3 soit divisible par 9; par suite $z=3z_1$, et z_1 est entier. Cela posé, nous ferons deux hypothèses, selon que z est impair ou pair.

1°. *Supposons z impair.* Alors $x-y$ et $x+y$ sont impairs; on a

$$x^3+y^3=(x+y)(x^2-xy+y^2)=(x+y)M,$$

et

$$4M=(x+y)^2+3(x-y)^2;$$

par conséquent, puisque $x+y$ est divisible par 3, M est aussi divisible par 3, mais non par une puissance supérieure; par conséquent, en désignant par a et b des nombres impairs, premiers entre eux, on doit poser

$$z_1=ab,\quad x+y=3^2Aa^3,\quad M=3b^3,$$

et, par suite

$$4b^3=(x-y)^2+3\left(\frac{x+y}{3}\right)^2.$$

D'ailleurs, b, diviseur de M, doit être de la forme f^2+3^2g, f et g étant premiers entre eux; on a ainsi

$$b=f^2+3g^2,\quad b^3=F^2+3G^2,\quad 4b^3=(F-3G)^2+3(F+G)^2;$$

et en identifiant les deux expressions de $4b^3$,

$$F+G=\frac{x+y}{3}=3Aa^3.$$

Mais le développement du cube de $f+g\sqrt{-3}$ donne

$$F=f(f^2-9g^2),\quad G=3g(f^2-g^2);$$

par suite

$$f(f^2-9g^2)+3g(f^2-g^2)=3Aa^3;$$

donc f serait divisible par 3, par suite b, et aussi x et y, que nous avons supposés premiers entre eux. Par conséquent, z ne peut être impair.

2°. *Supposons z pair;* on aurait

$$M = \left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2,$$

et, puisque x et y sont impairs, il en est de même de M. On doit donc poser

$$z_1 = 2ab, \quad x+y = 3^2.2^3.Aa^3, \quad M = 3b^3,$$

et, par suite

$$\left(\frac{x-y}{2}\right)^2 + 3\left(\frac{x+y}{6}\right)^2 = b^3.$$

Soient encore

$$b = f^2 + 3g^2, \quad b^3 = F^2 + 3G^2;$$

on en déduira

$$G = \frac{x+y}{6}, \quad \text{ou} \quad g(f^2 - g^2) = 4Aa^3.$$

D'ailleurs $f^2 + 3g^2$ et $f^2 + 3g^2 - 4g^2 = f^2 - g^2$ sont impairs; donc g est pair, et en désignant par α, β, γ trois nombres impairs et premiers entre eux dont le produit égale a, on doit poser

$$g = 4A\alpha^3, \quad f + g = \beta^3, \quad f - g = \gamma^3;$$

ou

$$g = 4\alpha^3, \quad f + g = A\beta^3, \quad f - g = \gamma^3.$$

On déduit de ces deux décompositions

$$\beta^3 - \gamma^3 = A(2\alpha)^3, \quad \text{ou} \quad \gamma^3 + (2\alpha)^3 = A\beta^3;$$

ces deux équations sont semblables à l'équation (1); on ramène donc l'équation proposée, dans laquelle l'une des inconnues contient le facteur 3^λ, à une autre semblable, dans laquelle l'une des inconnues ne contient plus que le facteur $3^{\lambda-1}$; en continuant de même, on ramènera l'équation proposée à une autre de même forme dans laquelle une des inconnues ne sera pas divisible par 3. Donc l'équation proposée est impossible lorsque A est égal à un nombre premier $p = 18n + 5$; ou au carré q^2 d'un nombre premier $q = 18n + 11$.

SECOND CAS. Considérons maintenant l'équation

$$x^3 + y^3 = 2^n A z^3,$$

dans laquelle A étant impair, le coefficient $2^n A$ représente l'un des quatre nombres $2p$, $2q^2$, $4p^2$, $4q$. Nous supposerons x, y, z entiers et premiers entre eux; x et y étant impairs. De plus, nous ferons deux hypothèses suivant que z est ou n'est pas divisible par 3.

1°. *Supposons z non divisible par* 3. On arrive facilement à l'équation

$$f(f^2 - 9g^2) = 2^{n-1}Aa^3;$$

mais $f^2 - 9g^2$ est impair, en même tempts que $b = f^2 + 3g^2$, et l'on a

$$f = 2^{n-1}A\alpha^3, \quad f + 3g = \beta^3, \quad f - 3g = \gamma^3;$$

ou bien

$$f = 2^{n-1}\alpha^3, \quad f + 3g = \beta^3, \quad f - 3g = \gamma^3.$$

Ces deux décomposition conduisent aux deux équations

$$\beta^3 + \gamma^3 = 2^n A\alpha^3, \quad \text{ou} \quad A\beta^3 + \gamma^3 = 2^n\alpha^3;$$

celles-ci sont impossibles suivant le module q, puisque, pour la première, les indéterminées α, β et γ ne sont pas divisible par 3.

2°. *Supposons z divisible par* 3. En posant $z = 3ab$, on arrive, connue plus haut, à l'équation

$$g(f^2 - g^2) = 2^{n-1}A\alpha^3,$$

et, puisque $f^2 - g^2$ est impair, à l'une des décompositions

$$g = 2^{n-1}A\alpha^3, \quad f + g = \beta^3, \quad f - g = \gamma^3;$$

ou bien

$$g = 2^{n-1}\alpha^3, \quad f + g = A\beta^3, \quad f - g = \gamma^3.$$

La seconde décomposition conduit à une équation déjà reconnue impossible; la première conduit à l'équation

$$\beta^3 - \gamma^3 = 2^n A\alpha^3.$$

Celle-ci est de même forme que la proposée; mais l'indéterminée du second membre contiendra un facteur 3 en moins. On conclura, connue précédemment, que l'équation proposée est impossible à résoudre en nombres entiers.

SECTION 5.

Les six valeurs générales de A données par M. SYLVESTER sont, jusqu'à présent, les seules valeurs connues qui rendent insoluble l'équation donnée, en ajoutant toutefois les valeurs

$$A = 1, 2, 3, 4, 18, 36,$$

données par FERMAT, EULER et LEGENDRE. On a encore le théorème suivant:

Pour que l'équation

$$X^3 + Y^3 = AZ^3,$$

soit vérifiée par des valeurs entières de X, Y, Z, A, il faut et il suffit que A appartienne à la forme

$$xy(x + y)$$

préalablement débarassée des facteurs cubiques qu'elle peut contenir.

En effet, on a l'identité

$$[x^3 - y^3 + 6x^2y + 3xy^2]^3 + [y^3 - x^3 + 6y^2x + 3yx^2]^3$$
$$= xy(x + y) \,.\, 3^3[x^2 + xy + y^2]^3,$$

et l'on résout l'équation proposée, par les valeurs

$$\begin{cases} X = x^3 - y^3 + 6x^2y + 3xy^2, \\ Y = y^3 - x^3 + 6y^2x + 3yx^2, \\ Z = 3(x^2 + xy + y^2), \\ A = xy(x+y). \end{cases}$$

Réciproquement, si l'équation est vérifiée pour les valeurs x_0, y_0, z_0 des variables, et si l'on pose

$$x = x_0^3, \quad y = y_0^3,$$

on a

$$xy(x+y) = A(x_0y_0z_0)^3.$$

C'est ce qu'il fallait démontrer. Il résulte encore de l'identité précédente que toute solution de l'équation proposée conduit à une série indéfinie d'autres solutions, en supposant A constant. Il faut excepter le cas de $x = \pm y$.

EXEMPLE: Pour $x = 1$, $y = 2$, on a la solution

$$17^3 + 37^3 = 6.21^3;$$

de laquelle on déduit une série indéfinie d'autres solutions. Ainsi l'équation

$$x^3 + y^3 = 6z^3,$$

est résoluble en nombres entiers, et d'une infinité de manières, bien que LEGENDRE ait affirmé le contraire.

PARIS, *Mai, 1879.*

Desiderata and Suggestions.

By Professor Cayley, *Cambridge, England.*

No. 4.—MECHANICAL CONSTRUCTION OF CONFORMABLE FIGURES.

Is it possible to devise an apparatus for the mechanical construction of conformable figures; that is, figures which are similar as regards corresponding infinitesimal areas? The problem is to connect mechanically two points P_1, P_2 in such wise that P_1 (1) shall have two degrees of freedom (or be capable of moving over a plane area) its position always determining that of P_2: (2) that if P_1, P_2 describe the infinitesimal lengths P_1Q_1, P_2Q_2, then the ratio of these lengths, and their mutual inclination, shall depend upon the position of P_1, but be independent of the direction of P_1Q_1: or what is the same thing, that if P_1 describe uniformly an indefinitely small circle, then P_2 shall also describe uniformly an indefinitely small circle, the ratio of the radii, and the relative position of the starting points in the two circles respectively, depending on the position of P_1.

Of course a pentagraph is a solution, but the two figures are in this case similar; and this is not what is wanted. Any unadjustable apparatus would give one solution only: the complete solution would be by an apparatus containing, suppose, a flexible lamina, so that P_1 moving in a given right line, the path of P_2 could be made to be any given curve whatever.

Cambridge, *July 9th, 1879.*

Notes.

I.

Note on Partitions.

By F. Franklin, *Fellow of the Johns Hopkins University.*

In a paper published in the Messenger of Mathematics (May, 1878), Prof. Sylvester has given a rule for abbreviating the calculation of $(w:i,j)-(w-1:i,j)$; where, to fix the ideas, let $(x:i,j)$ be regarded as the number of modes of composing x with j of the numbers $0, 1, 2, \ldots i$. The abbreviation consists in rejecting from the partitions of w all partitions whose highest number is not repeated and rejecting from the partitions of $w-1$ all partitions which do not contain i; the number of partitions thus rejected being shown to be the same in the two cases.

This becomes even more obvious if we convert the above (i,j) partitions into (j,i) partitions: that is, replace each of the above partitions by a corresponding one consisting of i of the numbers $0, 1, \ldots j$. This, as is well known, can be done by decomposing each number into a *column* of 1's and then recomposing by *rows*. Now, if we do this, it is plain that those partitions whose highest number was not repeated become partitions containing 1; and that those partitions which did not contain i become partitions having less than the full number of parts, or, in other words, partitions containing 0. So that Prof. Sylvester's abbreviation is equivalent to rejecting from the partitions of w those partitions which contain 1, and from the partitions of $w-1$ those which contain 0. And it is plain that the number of partitions of w which contain 1 is equal to the number of partitions of $w-1$ which contain 0; for the two sets of partitions are interchanged by the interchange of 0 and 1.

Obviously, instead of rejecting the partitions of w which contain 1 and those of $w-1$ which contain 0, we may reject the partitions of w which contain m (where m is any one of the numbers $1, 2, \ldots i$ (or j)) and those of $w-1$ which contain $m-1$; the reason being the same as above.

It may also be observed that the theorem in this form is easily obtained from the generating function. For $(w:i,j)-(w-1:i,j)$ is the coefficient of $a^w x^j$ in the development of

$$\frac{1-a}{(1-x)(1-ax)(1-a^2x)\ldots(1-a^ix)},$$

the numerator of which fraction may be written

$$1-a^m x - a\,(1-a^{m-1}x)\,,$$

so that $(w:i,j)-(w-1:i,j)$ is the difference between the coefficient of $a^w x^j$ in the development of

$$\frac{1-a^m x}{(1-x)(1-ax)\ldots(1-a^ix)};$$

and the coefficient of $a^{w-1}x^j$ in the development of

$$\frac{1-a^{m-1}x}{(1-x)(1-ax)\ldots(1-a^ix)};$$

and obviously if m is one of the numbers $1, 2, \ldots i$, these coefficients are, respectively, the number of the (i,j) partitions of w which do not contain m, and the number of the (i,j) partitions of $w-1$ which do not contain $m-1$: wherefore, the partitions of w which do contain m and those of $w-1$ which do contain $m-1$ may be simultaneously rejected.

II.

Some General Formulæ for Integrals of Irrational Functions.

By W. I. Stringham, *Fellow of the Johns Hopkins University.*

Let $\mathrm{Sh}\,z=\dfrac{ax+c}{\sqrt{b^2-c^2}}$, $\mathrm{Ch}\,z=\dfrac{aX+c}{\sqrt{b^2-c^2}}$. Since $\mathrm{Ch}^2 z-\mathrm{Sh}^2 z=1$, therefore

$$aX+c=\sqrt{a^2x^2+2acx+b^2}\,.$$

Let $\dfrac{c}{a}=h$, and $\dfrac{b^2-c^2}{a^2}=e^2$. Then

$$\int(x+h)^m(X+h)^n\,dx=e^{m+n+1}\int\mathrm{Sh}^m z\ \mathrm{Ch}^{n+1}z\,.\,dz\,.$$

If the exponent of $(x+h)$ is an odd integer, writing $(\mathrm{Ch}^2 z - 1)^m \,\mathrm{Sh}\, z$ for $\mathrm{Sh}^{2m+1} z$ and developing by the binomial theorem, we find at once

(1) $$\int (x+h)^{2m+1} (X+h)^{2n-1}\, dx$$

$$= \sum_0^m {}_p\, (-e^2)^{m-p} \frac{m!}{p!\,(m-p)!} \cdot \frac{(X+h)^{2n+2p+1}}{2n+2p+1}.$$

If the exponent of $(x+h)$ is an even integer, by means of the usual reduction formulæ for $\mathrm{Sh}^{2m} z\ \mathrm{Ch}^{2m} z$, it will be found that

(2) $$\int (x+h)^{2m} (X+h)^{2n-1}\, dx$$

$$= \frac{\left(n-\frac{1}{2}\right)!}{2\,(m+n)!} \sum_0^{n-1} {}_p\, e^{2p} \frac{(m+n-p-1)!}{\left(n-\frac{1}{2}-p\right)!} (x+h)^{2m+1} (X+h)^{2n-2p-1}$$

$$+ \frac{\left(m-\frac{1}{2}\right)!\,\left(n-\frac{1}{2}\right)!}{(m+n)!\,\left(\frac{1}{2}\right)!} \left(\sum_0^{m-1} {}_r\, (-)^r e^{2n+2r} \frac{(m-r-1)!}{\left(m-\frac{1}{2}-r\right)!} (x+h)^{2m-2r-1} (X+h) \right.$$

$$\left. + \frac{(-)^m}{\left(\frac{1}{2}\right)!} e^{2m+2n}\, \mathrm{Sh}^{-1} \frac{x+h}{e} \right).$$

Let $x+h-\mu = x+\lambda$. Then, by the binomial theorem,

$$(x+\lambda)^m (X+h)^n = \sum_0^m {}_p\, (-)^p \frac{m!\,\mu^p}{p!\,(m-p)!} e^{m+n-p}\, \mathrm{Sh}^{m-p} z\ \mathrm{Ch}^n z.$$

If the exponent of $(x+\lambda)$ is an even integer,

(3) $$\int (x+\lambda)^{2m} (X+h)^{2n-1}\, dx$$

$$= \sum_0^m {}_p\, \frac{(2m)!\,\mu^{2p}}{(2p)!\,(2m-2p)!} \frac{\left(n-\frac{1}{2}\right)!}{(m-p+n)!}$$

$$\times \left\{ \frac{1}{2} \sum_0^{n-1} {}_q\, \frac{(m-p+n-q-1)!}{\left(n-\frac{1}{2}-q\right)!} e^{2q} (x+h)^{2(m-p)+1} (X+h)^{2(n-q)+1} \right.$$

$$+\frac{\left(m-p-\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\left[\sum_{0}^{m-p-1}{}_{r}(-)^{r}\frac{(m-p-r-1)!}{\left(m-p-\frac{1}{2}-r\right)!}e^{2(n+r)}(x+h)^{2(m-p-r)-1}(X+h)\right.$$

$$\left.+\frac{(-)^{m-p}}{\left(\frac{1}{2}\right)!}e^{2(m-p+n)}\,\mathrm{Sh}^{-1}\frac{x+h}{e}\right]\Bigg\}$$

$$-\sum_{0}^{m}{}_{p}\frac{(2m)!\,\mu^{2p+1}}{(2p+1)!\,(2m-2p-1)!}\sum_{0}^{m-p-1}{}_{q}(-e^{2})^{m-p-q-1}\frac{(m-p-1)!}{q!\,(m-p-q-1)!}\frac{(X+h)^{2(n+q)+1}}{2(n+q)+1}.$$

If the exponent of $(x+\lambda)$ is an odd integer,

$$(4)\quad \int(x+\lambda)^{2m+1}(X+h)^{2n-1}\,dx$$

$$=\sum_{0}^{m}{}_{p}\frac{(2m+1)!\,\mu^{2p}}{(2p)!\,(2m-2p+1)!}\sum_{0}^{m-p}{}_{q}(-e^{2})^{m-p-q}\frac{(m-p)!}{q!\,(m-p-q)!}\frac{(X+h)^{2(n+q)+1}}{2(n+q)+1}$$

$$-\sum_{0}^{m}{}_{p}\frac{(2m+1)!\,\mu^{2p+1}}{(2p+1)!\,(2m-2p)!}\frac{\left(n-\frac{1}{2}\right)!}{(m-p+n)!}$$

$$\times\Bigg\{\frac{1}{2}\sum_{0}^{n-1}{}_{q}\frac{(m-p+n-q-1)!}{\left(n-\frac{1}{2}-q\right)!}e^{2q}(x+h)^{2(m-p)+1}(X+h)^{2(n-q)+1}$$

$$+\frac{\left(m-p-\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\left[\sum_{0}^{m-p-1}{}_{r}(-)^{r}\frac{(m-p-r-1)!}{\left(m-p-\frac{1}{2}-r\right)!}e^{2(n+r)}(x+h)^{2(m-p-r)-1}(X+h)\right.$$

$$\left.+\frac{(-)^{m-p}}{\left(\frac{1}{2}\right)!}e^{2(m-p+n)}\,\mathrm{Sh}^{-1}\frac{x+h}{e}\right]\Bigg\}.$$

When $\lambda=h$, μ vanishes, while for $p=0$, $\mu^{0}=1$, and the integrals (3) and (4) reduce to (2) and (1) respectively.

FEBRUARY *1st, 1879.*

III.

Note to the Article "On the Theory of Flexure," at page 13 (*Vol.* II) *of this Journal.*

By William H. Burr, *Rensselaer Polytechnic Institute.*

From the somewhat speculative nature of the article on the Theory of Flexure, resulting from the absence of experimental data on the "viscosity" of materials, it may be permissible to consider constant the intensity of stress in any section of a bent beam along a line parallel to the neutral axis of that section. Various considerations seem to indicate such a condition of stress. It is evident that that condition would accompany the greatest imaginable resistance which the beam could offer to external bending forces.

In order to represent this case for any beam not rectangular in section, it will only be necessary to put, in equation (74) of the article in question, consistently with the notation used in equation (46), $f(y')$ for each z_1 found in the parenthesis, and $2dy'$ for b. The general value for the bending moment then becomes

$$M = 4\frac{N_0}{\log z_1}\int_1^{y_1}\left(\frac{1}{4}f(y')^2\log\frac{f(y')^2}{e} - f(y')\log\frac{f(y')}{e} - \frac{3}{4}\right)dy'.$$

The value, for a rectangular section, will not be changed.

Troy, N. Y., *18 July, 1879.*

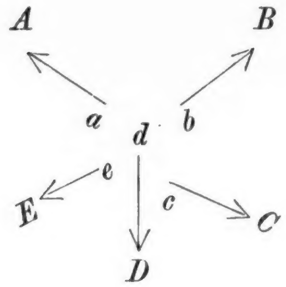
IV.

Generalization of Leibnitz's Theorem in Statics.

Extract of a Letter from Professor Crofton, *Royal Military Academy, Woolwich, to* Professor Sylvester.

. . . . . A small remark occurred to me the other day, which, it seems to me, can hardly have escaped notice, but no one that I know has met it. It is an extension of Leibnitz's Theorem in Statics: "If any number of forces

balance at a point, that point is the center of gravity of a system of equal particles at the extremities of the forces." Mine is, if any system of forces are in equilibrium, the center of gravity of the points of application a, b, c, d, e coincides with the center of gravity of the extremities of the forces A, B, C, D, E. This is an obvious consequence of Leibnitz's theorem.



On the Geographical Problem of the Four Colours.

By A. B. Kempe, B. A., *London, England.*

If we examine any ordinary map, we shall find in general a number of lines dividing it into districts, and a number of others denoting rivers, roads, etc. It frequently happens that the multiplicity of the latter lines renders it extremely difficult to distinguish the boundary lines from them. In cases where it is important that the distinction should be clearly marked, the artifice has been adopted by map-makers of painting the districts in different colours, so that the boundaries are clearly defined as the places where one colour ends and another begins; thus rendering it possible to omit the boundary lines altogether. If this clearness of definition be the sole object in view, it is obviously unnecessary that non-adjacent districts should be painted different colours; and further, none of the clearness will be lost, and the boundary lines can equally well be omitted, if districts which merely meet at one or two points be painted the same colour. (See Fig. 1.)

This method of definition may of course be applied to the case of any surface which is divided into districts. I shall, however, confine my investigations primarily to the case of what are known as simply or singly connected surfaces, *i. e.* surfaces such as a plane or sphere, which are divided into two parts by a circuit, only referring incidentally to other cases.

If, then, we take a simply connected surface divided in any manner into districts, and proceed to colour these districts so that no two adjacent districts shall be of the same colour, and if we go to work at random, first colouring as many districts as we can with one colour and then proceeding to another colour, we shall find that we require a good many different colours; but, by the use of a little care, the number may be reduced. Now, it has been stated somewhere by Professor De Morgan that it has long been known to map-makers as a matter of experience—an experience however probably confined to comparatively simple cases—that *four* colours will suffice in any case. That four colours may be necessary will be at once obvious on consideration of the case of one district surrounded by three others, (see Fig. 2), but that four colours will suffice in all cases is a fact which is by no means obvious, and has

rested hitherto, as far as I know, on the experience I have mentioned, and on the statement of Professor De Morgan, that the fact was no doubt true. Whether that statement was one merely of belief, or whether Professor De Morgan, or any one else, ever gave a proof of it, or a way of colouring any given map, is, I believe, unknown; at all events, no answer has been given to the query to that effect put by Professor Cayley to the London Mathematical Society on June 13th, 1878, and subsequently, in a short communication to the Proceedings of the Royal Geographical Society, Vol. I, p. 259, Professor Cayley, while indicating wherein the difficulty of the question consisted, states that he had not then obtained a solution. Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolour it throughout. After a somewhat arduous search, I have succeeded, suddenly, as might be expected, in hitting upon the weak point, which proved an easy one to attack. The result is, that the experience of the map-makers has not deceived them, the maps they had to deal with, viz: those drawn on simply connected surfaces, can, in every case, be painted with four colours. How this can be done I will endeavour—at the request of the Editor-in-Chief—to explain.

Suppose that we have the surface divided into districts in any way which admits of the districts being coloured with four colours, viz: blue, yellow, red, and green; and suppose that the districts are so coloured. Now if we direct our attention to those districts which are coloured red and green, we shall find that they form one or more detached regions, *i. e.* regions which have no boundary in common, though possibly they may meet at a point or points. These regions will be surrounded by and surround other regions composed of blue and yellow districts, the two sets of regions making up the whole surface. It will readily be seen that we can interchange the colours of the districts in one or more of the red and green regions without doing so in any others, and the map will still be properly coloured. The same remarks apply to the regions composed of districts of any other pair of colours. Now if a region composed of districts of any pair of colours, say red and green as before, be of either of the forms shown in Figures 3 and 4, it will separate the surface into two parts, so that we may be quite certain that no yellow or blue districts in one part can belong to the same yellow and blue region as any yellow or blue district in the other part. Thus any specified blue district, for example, in one part can, by an interchange of the colours in the yellow and

blue region to which it belongs, be converted into a yellow district, whilst any specified yellow district in the other part remains yellow.

Now let us consider the state of things at a point where three or more boundaries and districts meet. It will be convenient to term such a point a *point of concourse.* If three districts meet at the point, they must be coloured with three different colours. If four, they may be coloured with two or three colours only in some cases, but on the other hand they may be coloured with four, as in Fig. 5. If the districts a and c in this case belong to different red and green regions, we can interchange the colours of the districts in one of these regions, and the result will be that the districts a and c will be of the same colour, both red or both green. If a and c belong to the same red and green region, that region will form a ring as in Fig. 4, and b will be in one of the parts into which it divides the surface and d in the other, so that the yellow and blue region to which b belongs, will be different from that to which d belongs; if, therefore, we interchange the colours in either of these regions, b and d will be of the same colour, both yellow or both blue. Thus we can always reduce the number of colours which meet at the point of concourse of four boundaries to three.

The same thing may be shown in the case of points of concourse where five boundaries meet. The districts meeting at the point may happen to be coloured with only three colours, but they may happen to be coloured with four. Fig. 6 shows the only form which the colouring can take in that case, one colour of course occurring twice. If a and c belong to different yellow and red regions, interchanging the colours in either, a and c become both yellow or both red. If a and c belong to the same yellow and red region, see if a and d belong to different green and red regions; if they do, interchanging the colours in either region, a and d become both green or both red. If a and c belong to the same yellow and red region, and a and d belong to the same green and red region, the two regions cut off b from e, so that the blue and green region to which b belongs is different from that to which d and e belong, and the blue and yellow region to which e belongs is different from that to which b and c belong. Thus, interchanging the colours in the blue and green region to which b belongs, and in the blue and yellow region to which e belongs, b becomes green and e yellow, a, c and d remaining unchanged. In each of the three cases the number of colours at the point of concourse is reduced to three.

It will be unnecessary for my purpose to take the case of a larger number of boundaries. Later on, we shall see that we can arrange the colours so

that not only will three colours only meet at any given point of concourse, however many boundaries meet there, but also at no point of concourse in the map will four colours appear. It is, however, at present, enough, (and I have proved no more), that if less than six boundaries meet at a point we can always rearrange the colours of the districts so that the number of colours at that point shall only be three.

Before leaving this part of the investigation, I may point out that it does not apply to the case of other surfaces. A glance at Fig 7, which represents an anchor ring, will show that a ring-shaped district, a a, if it *clasps* the surface, does not divide it into two parts, so that the foregoing proof fails. In fact, six colours may be required to colour an anchor ring. For, if two *clasping* boundaries be described so as to divide the ring into two bent cylindrical portions, and if each portion be divided into three parts by longitudinal boundaries, a, b, c being the three parts on one and d, e, f being those on the other, so that a abuts on d and e at one end, and on e and f at the other; b abuts on e and f at one end, and on f and d at the other; c abuts on f and d at one end, and on d and e at the other, then a, b, c, d, e, f must all be of different colours.

Returning to the case of the simply connected surface, and putting aside for the moment the question of colouring, let us consider some points as to the structure of the map on its surface. This map can have in it *island-districts* having one boundary (Fig. 8); and *island-regions* (Fig. 9) composed of a number of districts; also, *peninsula-districts*, having one boundary and one point of concourse (Fig. 10); and *peninsula-regions* (Fig. 11); *complex-districts*, which have islands and peninsulas in them; and *simple-districts* which have none, and have as many boundaries as points of concourse (Fig. 12). It should also be noticed that, with the exception of those boundaries which are endless, such as that in Fig. 8, and those which have one point of concourse such as that in Fig. 10, every boundary ends in two points of concourse; and further, that every boundary belongs to two districts.

Now, take a piece of paper and cut it out to the same shape as any simple- island- or peninsula-district, but rather larger, so as just to overlap the boundaries when laid on the district. Fasten this *patch* (as I shall term it) to the surface and produce all the boundaries which meet the patch, (if there be any, which will always happen except in the case of an island), to meet at a point, (a point of concourse) within the patch. If only two boundaries meet the patch, which will happen if the district be a peninsula, join them across

the patch, no point of concourse being necessary. The map will then have one district less, and the numbers of boundaries will also be reduced. Fig. 13 shows the district before the patch is put on, the place where it is going to be, being indicated by the dotted line, and Fig. 14 shows what is seen after the patch (again denoted by the dotted line) has been put on, and the boundaries have been produced to meet in a point on it. This patching process can be repeated as long as there is a simple district left to operate upon, the patches being in some cases stuck partially over others. If we confine our operations to an island or peninsula, we shall at length get rid of the island or peninsula, and doing this in the case of all the islands and peninsulas, complex-districts will be reduced to simple ones, and can be got rid of by the same process. We can thus, by continually patching, at length get rid of every district on the surface, which will be reduced to a single district devoid of boundaries or points of concourse. The whole map is patched out.

Now, reverse the process, and strip off the patches in the reverse order, taking off first that which was put on last, as each patch is stripped off it discloses a new district, and the map is *developed* by degrees.

Suppose that at any stage of this development, when we have stripped off a number of patches, there are on the surface

D districts
B boundaries
P points of concourse,

and suppose that after the next patch is stripped off there are

D' districts
B' boundaries
P' points of concourse.

If the patch has no point of concourse on it or line, *i. e.*, if when it is stripped off an island is disclosed,

$$P' = P$$
$$D' = D + 1$$
$$B' = B + 1.$$

If the patch has no point of concourse but only a single line, so that when it is stripped off a peninsula is disclosed,

$$P' = P + 1$$
$$D' = D + 1$$
$$B' = B + 2.$$

If the patch has a point of concourse on it where σ boundaries meet

$$P' = P + \sigma - 1$$
$$D' = D + 1$$
$$B' = B + \sigma.$$

In each case therefore

$$P' + D' - B' - 1 = P + D - B - 1,$$

i. e., at every stage of the development

$$P + D - B - 1$$

has the same value. But at the first stage

$$P = 0$$
$$D = 1$$
$$B = 0.$$

Therefore we always have

$$P + D - B - 1 = 0^{*}. \quad \ldots\ldots\ldots \quad (1)$$

That is *in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of boundaries*†.

Let d_1, d_2, d_3, etc., denote the number of districts at any stage, which have one, two, three, etc., boundaries, so that

$$D = d_1 + d_2 + d_3 + \ldots,$$

and let p_3, p_4, etc., denote the number of points of concourse, at the same stage of the development, at which three, four, etc., boundaries meet, so that

$$P = p_3 + p_4 + \ldots.$$

Then, since every boundary belongs to two districts,

$$2B = d_1 + 2d_2 + 3d_3 + \ldots,$$

and since every boundary ends in two points of concourse, except in the case of continuous boundaries which have no points of concourse, of which let there be β_0, and boundaries round peninsula districts which have one point of concourse, of which let there be β_1, therefore,

$$2B = 2\beta_0 + \beta_1 + 3p_3 + 4p_4 + \ldots.$$

Thus, since (1) may be written

$$(6D - 2B) + (6P - 4B) - 6 = 0,$$

we have $$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - \text{etc.} = 0,$$

the first five terms being the only positive ones. At least one, therefore, of the quantities $d_1 . d_2 . d_3 . d_4 . d_5$ must not vanish, *i. e. every map drawn on a simply connected surface must have a district with less than six boundaries.*

* The formula (1) was first stated as connecting the number of angular points, faces, and edges of a polyhedron by Cauchy. † See note following this paper.

It may readily be seen that this proof applies equally well to an *island-region* or *peninsula-region* as to the whole map. The result is, that we can patch out any simply-connected map, never putting a patch on a district with more than five boundaries. Consequently, if we develop a map so patched out, since each patch, when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at the point of concourse on the patch*. Of course districts which, when first disclosed, have only five boundaries may ultimately have thousands.

Returning to the question of colour, if the map at any stage of its development, can be coloured with four colours, we can arrange the colours so that, at the point of concourse on the patch next to be taken off, where less than six boundaries meet, only three colours shall appear, and, therefore, when the patch is stripped off, only three colours surround the disclosed district, which can, therefore, be coloured with the fourth colour, *i. e.* the map can be coloured at the next stage. But, at the first stage, one colour suffices, therefore, four suffice at all stages, and therefore, at the last. This proves the theorem and shows how the map may be coloured.

I stated early in the article that I should show that the colours could be so arranged that only three should appear at every point of concourse. This may readily be shown thus: Stick a small circular patch, with a boundary drawn round its edge, on every point of concourse, forming new districts. Colour this map. Only three colours can surround any district, and therefore the circular patches. Take off the patches and colour the uncovered parts the same colour as the rest of their districts. Only three colours surrounded the patches, and therefore only three will meet at the points of concourse they covered.

A practical way of colouring any map is this, which requires no patches. Number the districts in succession, always numbering a district which has less than six boundaries, not including those boundaries which have a district already numbered on the other side of them. When the whole map is numbered, beginning from the highest number, letter the districts in succession with four letters, $a.b.c.d$, rearranging the letters whenever a district has four round it, so that it may have only three, leaving one to letter the district with. When the whole map is lettered, colour the districts, using different colours for districts lettered differently.

* See note following this paper.

Two special cases should be noticed.

(1). If, excluding island and peninsula districts from the computation, every district is in contact with an even number of others along every circuit formed by its boundaries, three colours will suffice to colour the map.

(2). If an even number of boundaries meet at every point of concourse, two colours will suffice. This species of map is that which is made by drawing any number of continuous lines crossing each other and themselves any number of times.

If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering the points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. Following this up, we may ask what are the linkages which can be similarly lettered with not less than n letters?

The classification of linkages according to the value of n is one of considerable importance. I shall not, however, enter here upon this question, as it is one which I propose to consider as a part of an investigation upon which I am engaged as to the general theory of linkages. It is for this reason also that I have preferred to treat the question discussed in this paper in the manner I have done, instead of dealing with the analogous linkage.

I will conclude with a theorem which can readily be obtained as a corollary to the preceding results. It is one of which I long endeavoured to obtain an independent proof, as a means of solving the four-colour problem. The polyhedra mentioned are to be understood to be simply connected ones. The theorem is this:

"*Polyhedra can be added to the faces of any polyhedron so that in the resulting polyhedron* (1) *the faces are all triangles,* (2) *the number of edges meeting at every angular point is a multiple of three.*

JUNE *23d, 1879.*

Fig. 1.

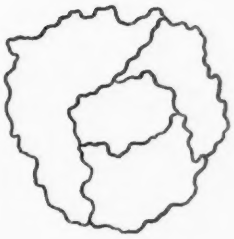


Fig. 2.

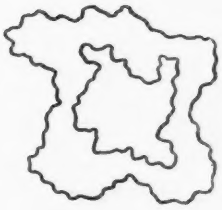


Fig. 3.

Fig. 4.

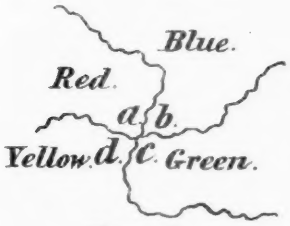


Fig. 5.

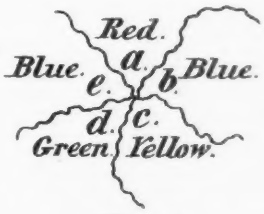


Fig. 6.

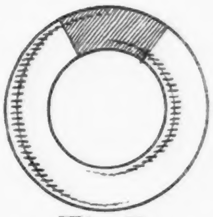


Fig. 7.

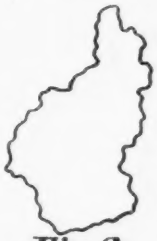


Fig. 8.

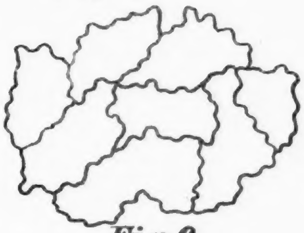


Fig. 9.

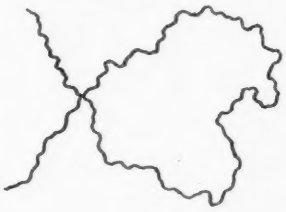


Fig. 10.

Fig. 11.

Fig. 12.

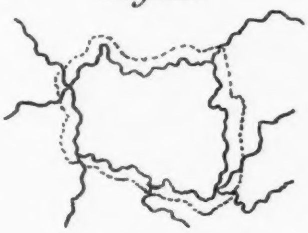


Fig. 13.

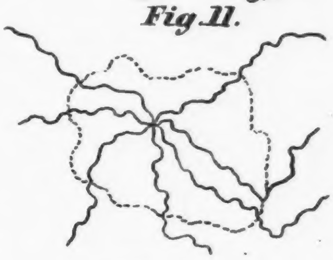


Fig. 14.

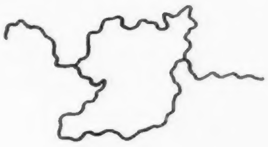


Fig. 15.

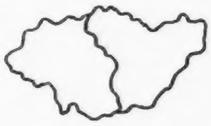


Fig. 16.

KEMPE, Geographical Problem.

A. HOEN & CO. LITH. BALTIMORE.

Note on the Preceding Paper.

By William E. Story.

In the foregoing valuable paper on the "*Geographical Problem of the Four Colours,*" Mr. Kempe has substantially proved the fundamental theorem, which has been so long a desideratum, by a very ingenious method; but it seems desirable, to make the proof absolutely rigorous, that certain cases which are liable to occur, and whose occurrence will render a change in the formulae, as well as some modification of the method of proof, necessary, should be considered separately, and I have endeavoured to do this in the following note.

1. In the notation used on page 197, if, at any stage of the development, the patch next to be stripped off has no point of concourse or line on it, when it is stripped off an island will be disclosed, and

$$P' = P, \quad D' = D + 1, \quad B' = B + 1.$$

If the patch has no point of concourse but only a single line, when it is stripped off either a peninsula or a district with two boundaries (as in Fig. 15) will be disclosed; in the first case we have

$$P' = P + 1, \quad D' = D + 1, \quad B' = B + 2,$$

and in the second case

$$P' = P + 2, \quad D' = D + 1, \quad B' = B + 3.$$

These formulae hold only if the boundaries joined by the line on the patch counted as *two* (and not *one*, as in Figs. 16 and 1) before the patch was put on. If the patch have a point of concourse in which σ boundaries meet, and if the district disclosed when the patch is stripped off have β boundaries, we have

$$P' = P + \beta - 1, \quad D' = D + 1, \quad B' = B + \beta.$$

These equations are identical with those at the top of page 198, *i. e.* $\beta = \sigma$, only when three and no more boundaries meet in each point of concourse about the district patched out, for σ is evidently the number of boundaries meeting the boundaries of this district, each of the former counting for one or two according as it meets the latter in one or two *different* points.

In either of the above cases
$$P' + D' - B' - 1 = P + D - B - 1. \quad . \quad . \quad . \quad . \quad . \quad (a)$$
But if the patch has no point of concourse but only a single line *forming part of the boundary of an island-district* on the patched map, so that when the patch is stripped off one of the forms of Figs. 16 and 1 is disclosed, we have for the former
$$P' = P + 2, \quad D' = D + 1, \quad B' = B + 2,$$
and for the latter (if the point in which the *two* boundaries meet be called a point of concourse)
$$P' = P + 1, \quad D' = D + 1, \quad B' = B + 1;$$
and hence for either of these cases
$$P' + D' - B' - 1 = P + D - B, \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$
instead of (a).

We will define a *contour* as an aggregate of boundaries such that, while any two of them are mutually connected, either directly or by means of other boundaries of the same contour, they are not connected with any other boundaries in the map. Such a contour will be *simple* or *complex* according as it consists of *one* or *more than one* boundary. Each contour may be considered as forming a map by itself, which may be coloured accordingly. In the process of patching out by Mr. Kempe's method the map formed by any *complex-contour*, we must arrive sooner or later at one of the forms of Figs. 16 and 1, next at an island, and then this disappears. In the reverse process we have at the first stage
$$P = 0, \quad D = 1, \quad B = 0,$$
i. e.
$$P + D - B - 1 = 0;$$
at the second stage, by (a),
$$P + D - B - 1 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
at the third stage, by (b),
$$P + D - B - 1 = 1;$$
and at every subsequent stage, by (a),
$$P + D - B - 1 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
Of course, in the case of a map formed by a *simple-contour* only the first and second stages exist, and for such a map equation (1) holds. If then x of the contours formed by the boundaries of any map are complex, for that map
$$P + D - B - 1 = x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
That is *in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of*

boundaries and number of complex-contours together. This takes the place of the theorem at the middle of page 198.

It is possible, as we have seen, that two boundaries of themselves form a point of concourse, as in Fig 1. Hence to the series $p_3, p_4, \ldots$ should be prefixed p_2, the number of points of concourse at the given stage of the development at which two boundaries meet.* Then

$$P = p_2 + p_3 + p_4 + \ldots$$

and

$$2B = 2\beta_0 + \beta_1 + 2p_2 + 3p_3 + 4p_4 + \ldots$$

Thus, since (3) may be written

$$(6D - 2B) + (6P - 4B) - 6(\varkappa + 1) = 0,$$

we have

$$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - \ldots = 0,$$

as at the bottom of page 198.

2. At the top of page 199 the statement is made: "*if we develop a map so patched out, since each patch, when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at the point of concourse on the patch.*" Now, as we have noticed above, the number of boundaries meeting in the point of concourse on a patch is equal to the number of boundaries of the district covered by the patch only when the number of boundaries meeting in each point of concourse about the district does not exceed three, and the statement quoted is therefore true only in this case. This difficulty may be obviated as follows:

Any point of concourse in which more than three boundaries meet may be removed, or replaced by a number of points of concourse in which only three boundaries meet, by sticking on it a small circular patch with a line drawn round it, of which those portions have been erased, which form the boundaries between the patch and any *one* of the districts bordering on it. Such a patch we will designate an *auxiliary-patch.* The number of districts in any map is not altered by an auxiliary-patch, since this forms only an extension of one of its districts over a point of concourse.

Having first *modified* the map by sticking an auxiliary-patch on each point of concourse in which more than three boundaries meet, we proceed to patch out the modified map as above, always putting a patch on a district with less than six boundaries, and sticking an auxiliary-patch on every point

* A point of concourse in which two boundaries meet counts for *one* in p_2 and also for *two* in β_1.

of concourse in which four or five boundaries meet, as it is produced (in this process a point of concourse in which more than five boundaries meet is never produced, nor are two auxiliary patches ever put on in immediate succession after the first district has been patched out). We thus finally arrive at a map containing one district and no boundary, which we colour with either one of four colours. Then, developing the map by stripping off the patches (including auxiliary patches) in the order inverse to that in which they were put on, we colour each district as it is disclosed. We will suppose that at a stage of the development, in which a certain district was disclosed, the map has been coloured with the four colours. The patch next to be stripped off will be either an ordinary patch with no line, or with one line and no point of concourse, or with a point of concourse in which not more than three boundaries meet; or it will be an auxiliary-patch. If it is an ordinary patch, when it is stripped off a district will be disclosed, on which border not more than three other districts, and at least one of the four colours will thus be at our disposal for the new district. If it is an auxiliary-patch, when it is stripped off a point of concourse will be disclosed, in which four or five boundaries, and not more than five districts, meet. The colours of these districts are to be extended over their uncovered portions, and the number of colours at the point of concourse reduced to not more than three by the method of page 195. The next patch (that on which is the point of concourse lately covered by the auxiliary-patch) will thus disclose, when stripped off, a district with four or five boundaries, surrounded by not more than five districts of not more than three different colours, and for which therefore at least one of the four colours will be at our disposal. Thus the map will be coloured with the four colours.

It is evidently unnecessary to extend the colours of the adjacent districts up to the point of concourse disclosed by an auxiliary-patch until we have disclosed the whole map as first *modified*, for a point of concourse appearing at an earlier stage will lie on a patch which has to be stripped off at the next stage.

BALTIMORE, *August 22d, 1879.*

The Quaternion Formulae for Quantification of Curves, Surfaces and Solids, and for Barycentres.

By W. I. Stringham, *Fellow of the Johns Hopkins University.*

In order to avoid a clumsy circumlocution, I have ventured to use the word quantification to denote in general that class of operations expressed in the several special cases by the terms, rectification, quadrature and cubature. Some of the quaternion formulae for quantification were given in a paper entitled "Investigations in Quaternions," communicated to the *American Academy of Arts and Sciences*, 9th January, 1878. They are here reproduced in a more general form together with the formulae for barycentres.

Quantification.

Let dM represent in general the element of arc, surface, or volume. Then if $\rho = \psi(t)$ be the vector equation of any curve in space, where t is a scalar variable,

$$dM = ds = \mathrm{T}\rho'.dt, \tag{1}$$

where $\rho' = \mathrm{D}_t\rho =$ the tangent to the curve at the point ρ. The formula for rectification, therefore, is

$$s = \int \mathrm{T}\rho'.dt. \tag{2}$$

The double area of the triangle, two of whose sides are ρ and ρ', is $\mathrm{TV}\rho\rho'$, and the element of area swept by ρ is

$$dM = dA = \frac{1}{2}\,\mathrm{TV}\rho\rho'.dt. \tag{3}$$

Hence for quadrature of plane areas

$$A = \frac{1}{2}\int \mathrm{TV}\rho\rho'.dt. \tag{4}$$

This formula is sufficient to determine the area of a sector whose vertex is at the origin. In order to determine the area of any other sector (the surface being plane) whose vertex is δ, it is only necessary to write $\pi = \rho - \delta$ instead of ρ, δ being the vector of the new origin with reference to the old. Thus, to

determine an area limited by a given chord, make δ the vector of one extremity of the chord and determine the limits of integration by assuming $\rho=\delta$ and $\rho=\delta+\eta$, where η is the chord in question.

A surface in general is represented by the vector equation $\rho=\chi(t,u)$, where t, u, are independent scalar variables. If at any instant u remain constant and t vary, ρ will describe a determinate arc upon the surface; if t remain constant and u vary, ρ will describe another arc cutting the former at the point ρ. Hence

$$\rho'_1=D_t\rho\,,\quad \rho'_2=D_u\rho \tag{5}$$

will represent two tangents to the surface intersecting at their common point of contact. Therefore,

$$dM=dS=TV\rho'_1\rho'_2.dtdu \tag{6}$$

will represent an elementary parallelogram upon the surface. Hence, for quadratures in general,

$$S=\int TV\rho'_1\rho'_2.dt,u, \tag{7}$$

where dt,u means $dtdu$. The equation $\rho=\chi(t,u)$, for the special case of plane surfaces, may be written $\rho=u\psi(t)=u\tau$, the origin being on the surface. The formula last written then becomes

$$S=\frac{1}{2}\,[u^2]_u^1\int TV\tau\tau'.dt, \tag{8}$$

where $\tau'=D_t\tau$. This formula is a more general expression for (4) wherein the limits for u were 0 and 1. It is evident that the choice of any other limits for u would determine an area lying between two similar curves.

If, in the equation of a plane curve, $\rho=\psi(t)$, ϕ be the angle of revolution, and the curve be revolved about an axis α, (with the condition $T\alpha=1$), then the element of area of the surface of revolution will be

$$dM=dS=TV\alpha\rho.d\phi.T\rho'.dt, \tag{9}$$

whence

$$S=[\phi]_0^{\phi}\int T\rho'V\alpha\rho.dt, \tag{10}$$

where $\rho'=D_t\rho$ as before.

Let now $\omega=z\rho$, where $\rho=\chi(t,u)$ and z, t, u are independent scalar variables. When z varies between 0 and 1, the extremity of ω generates the solid enclosed by the surface whose vector generator is ρ. Since $S\omega'_1\omega'_2\omega'_3$ is the volume of the parallelopiped three of whose edges are ω'_1, ω'_2, ω'_3, then if

$$D_t\omega=\omega'_1\,,\quad D_u\omega=\omega'_2\,,\quad D_z\omega=\omega'_3\,,$$

an elementary parallelopiped within the solid will be represented by

$$dM=dV=S\omega'_1\omega'_2\omega'_3.dt,u,z. \tag{11}$$

Hence, for cubatures in general,

$$V=\int\int\int \mathrm{S}\omega'_1\omega'_2\omega'_3.dt,u,z, \tag{12}$$

and since $\omega'_1=z\rho'_1$, $\omega'_2=z\rho'_2$, $\omega'_3=\rho$,

$$\therefore\ V=\frac{1}{3}[z^3]_z^1\int\int \mathrm{S}\rho\rho'_1\rho'_2.dt,u, \tag{13}$$

where $\rho'_1=\mathrm{D}_t\rho$, $\rho'_2=\mathrm{D}_u\rho$. This formula enables us to determine the contents of sectors, or cones whose vertices are at the origin. For the cubature of a portion of the solid limited by a plane, instead of ρ write $\rho-\delta$, where δ is the vector of some point in the given plane, and determine the limits of integration by making ρ satisfy the equation of the plane in question.

The equation, $\rho=u\psi(t)$, of a plane surface, with the condition that the surface be revolved about an axis α, will be sufficient to determine all points of a space enclosed by a surface of revolution. If $\mathrm{T}\alpha=1$ and ϕ be the angle of revolution, $\mathrm{V}\alpha\rho$ is the projection of ρ on the normal to the plane of $\alpha\rho$ and $\mathrm{V}\alpha\rho\,.\,d\phi$ is identical with what ρ'_2du becomes in this case, that is, it is the tangent to a parallel of latitude. Then

$$dM=dV=\mathrm{S}\rho\rho'\mathrm{V}\alpha\rho\,.\,z^2dtdzd\phi, \tag{14}$$

$\rho'=\mathrm{D}_t\rho$. Hence, for solids of revolution,

$$V=\frac{1}{3}[z^3\phi]_z^1\,\big|_0^\phi\int \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho\,.\,dt. \tag{15}$$

The order of substitution of limits is $1, z$, afterwards $\phi, 0$. This may also be written

$$V=\frac{1}{3}[z^3\phi]_z^1\,\big|_0^\phi\int \mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho\,.\,dt; \tag{16}$$

for $\mathrm{S}\rho\rho'\mathrm{V}\alpha\rho=\mathrm{SV}\rho\rho'\mathrm{V}\alpha\rho$, and $\mathrm{V}(\mathrm{V}\rho\rho'\mathrm{V}\alpha\rho)=0$; therefore,

$$\mathrm{S}^2\mathrm{V}\rho\rho'\mathrm{V}\alpha\rho=\mathrm{T}^2\mathrm{V}\rho\rho'\mathrm{V}\alpha\rho,\quad\text{or}\quad \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho=\pm\,\mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho.$$

In the formulae for cubatures, the limits 0 and 1 for z give the contents of the space swept by the generating vector of the surface; any other limits would determine the contents of a space lying between two similar surfaces, *i. e.* of a shell.

Barycentres.

In Hamilton's equation*

$$\Sigma m_r(e_r-e)=0,\quad\text{or}\quad \Sigma m_r\epsilon_r=0, \tag{17}$$

wherein the e and e_r are symbols of position in space, the ϵ_r are vectors to the points e_r from e, and the m_r are scalar quantities, or weights, with which the

* Elements of Quaternions, p. 89.

points e_r may be regarded as being loaded,—the point e is defined as the barycentre of the system.

Of the system represented by this equation the barycentre is evidently at the origin of vectors. A change of origin, referred to which $\bar{\gamma}$ is the vector to the barycentre, gives

$$\Sigma m_r(\varepsilon_r - \bar{\gamma}) = 0,$$

that is,

$$\bar{\gamma} = \frac{\Sigma m_r \varepsilon_r}{\Sigma m_r}. \tag{18}$$

For a continuous, homogeneous mass, this equation assumes the form

$$\bar{\gamma} = \int \rho . dM \div \int dM, \tag{19}$$

where dM is the element of mass. It remains to substitute in this formula the values already obtained for dM in the several cases above considered.

For arcs $dM = \mathrm{T}\rho'.dt$, (3), and

$$\bar{\gamma} = \int \rho \mathrm{T}\rho'.dt \div \int \mathrm{T}\rho'.dt. \tag{20}$$

For surfaces in general $dM = \mathrm{TV}\rho'_1\rho'_2.dt,u$, (6), and

$$\bar{\gamma} = \int\int \rho \mathrm{TV}\rho'_1\rho'_2.dt,u \div \int\int \mathrm{TV}\rho'_1\rho'_2.dt,u. \tag{21}$$

For plane surfaces (8) this becomes

$$\bar{\gamma} = \frac{1}{3}[u^3]^1_u \int \tau \mathrm{TV}\tau\tau'.dt \div \frac{1}{2}[u^2]^1_u \int \mathrm{TV}\tau\tau'.dt, \tag{22}$$

$u\tau = \rho, \tau' = \mathrm{D}_t\tau$. For surfaces of revolution dM is (9), and

$$\bar{\gamma} = \int \rho \mathrm{T}\rho'\mathrm{V}\alpha\rho.dt \div \int \mathrm{T}\rho'\mathrm{V}\alpha\rho.dt, \tag{23}$$

$\alpha =$ axis of revolution, $\mathrm{T}\alpha = 1$. For solids in general dM is (11), and

$$\begin{aligned}\bar{\gamma} &= \int\int\int \omega \mathrm{S}\omega'_1\omega'_2\omega'_3.dt,u,z \div \int\int\int \mathrm{S}\omega'_1\omega'_2\omega'_3.dt,u,z \\ &= \frac{1}{4}[z^4]^1_z \int\int \rho \mathrm{S}\rho\rho'_1\rho'_2.dt,u \div \frac{1}{3}[z^3]^1_z \int\int \mathrm{S}\rho\rho'_1\rho'_2.dt,u,\end{aligned} \tag{24}$$

$\rho'_1 = \mathrm{D}_t\rho$, $\rho'_2 = \mathrm{D}_u\rho$. For solids of revolution dM is (14), and

$$\bar{\gamma} = \frac{1}{4}[z^4]^1_z \int \rho \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho.dt \div \frac{1}{3}[z^3]^1_z \int \mathrm{S}\rho\rho'\mathrm{V}\alpha\rho.dt, \tag{25}$$

$\rho' = \mathrm{D}_t\rho$. This last formula [see (16)] may also be written

$$\bar{\gamma} = \frac{1}{4}[z^4]^1_z \int \rho \mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho.dt \div \frac{1}{3}[z^3]^1_z \int \mathrm{TV}\rho\rho'\mathrm{V}\alpha\rho.dt. \tag{26}$$

It is to be noticed that the formulae (24), (25), (26) give the barycentres of the entire solids, or of shells, according as the limits assumed for z are 0

and 1, or some other. Suppose the limits to be z and $z+\varepsilon$, where ε is infinitesimal. If the terms containing the second and higher powers of ε be neglected, there will be in the numerator the factor $(z+\varepsilon)^4 - z^4 = 4z^3\varepsilon$, and in the denominator, $(z+\varepsilon)^3 - z^3 = 3z^2\varepsilon$; and (24) will reduce to

$$\bar{\gamma} = z\iint \rho S\rho\rho'_1\rho'_2 . dt, u \div \iint S\rho\rho'_1\rho'_2 . dt, u, \tag{27}$$

the barycentric vector to an infinitely thin shell. The relative thickness of the shell, as shown by the factor z in the equation, $\omega = z\chi\,(t, u)$, of the shell, will be determined by the position of the origin of the generating vector.

Applications.

The ellipsoid will afford a convenient illustration of the application of the above methods. Its equation is

$$\rho = \alpha \cos x + \sigma \sin x,$$

where

$$\sigma = \beta \cos y + \gamma \sin y,$$

and α, β, γ are the three principal semiaxes. By differentiation and reduction

$$TV\rho'_1\rho'_2 = as \sin x \sqrt{1 - \sin^2 w \cos^2 x},$$

where $a = T\alpha$, $b = T\beta$, $c = T\gamma$, $s = T\sigma'$, and $\sin^2 w = 1 - \frac{b^2c^2}{a^2s^2}$, or $\cos w = \frac{bc}{as}$. Hence by (7)

$$S = \int \frac{bc}{\cos w} \int \sin x \sqrt{1 - \sin^2 w \cos^2 x} . dx, y.$$

Let $\sin v = \sin w \cos x$, $\cos v = \sqrt{1 - \sin^2 w \cos^2 x}$. Then

$$S = \int \frac{2bc}{\sin 2w} \int \cos^2 v . dv, y = \int \frac{bc}{2} \frac{[2v + \sin 2v]_0^v}{\sin 2w} . dy. \tag{28}$$

When the surface is one of revolution, say the prolate ellipsoid, then $b = c = s$, $\sin^2 w = \frac{a^2 - b^2}{a^2}$, and the expression (28) reduces to

$$S = \frac{a^2 b}{4\sqrt{a^2 - b^2}} [y\,(2v + \sin 2v)]_0^v\big|_0^y.$$

The order of substitution of limits is v, 0, afterwards y, 0.

By easy transformations it will be found that $V\rho\rho'_1 = V\alpha\sigma$, $V\sigma\sigma' = V\beta\gamma$, and therefore

$$S\rho\rho'_1\rho'_2 = abc \sin x.$$

Hence, for the ellipsoidal shell [see (13)],

$$V = \frac{abc}{3} [z^3]_z^1 \iint \sin x . dx, y = \frac{abc}{3} [z^3 y \cos x]_0^x\big|_0^y\big|_z^1.$$

The order of substitution of limits is x, 0; y, 0; 1, z.

The value of $\rho S\rho\rho'_1\rho'_2$ is $abc\,(\alpha \sin x \cos x + \sigma \sin^2 x)$, and its complete integral is

$$\frac{abc}{4}\left[2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\right]_0^x\Big|_0^y .$$

Hence the barycentric vector of the ellipsoidal shell is

$$\bar{\gamma} = \frac{3\left[z^4\{2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\}\right]_0^x\Big|_0^y\Big|_z^1}{16\left[z^3 y \cos x\right]_0^x\Big|_0^y\Big|_z^1} .$$

The order of substitution of limits is important; it is as above in the expression for V. If the shell be infinitely thin, in this case an ellipsoidal Chaslesian shell, the application of formula (27) gives

$$\bar{\gamma} = \frac{z\left[2\alpha y \sin^2 x + (\beta \sin y - \gamma \cos y)(2x - \sin 2x)\right]_0^x\Big|_0^y}{4\left[y \cos x\right]_0^x\Big|_0^y} .$$

For the ellipsoidal solid $\bar{\gamma}$ is the expression last written multiplied by $\frac{3}{4z}$. The barycentric vector to the half solid, bounded by the plane of $\alpha\beta$, will easily be found to have the value $\frac{3}{8}\gamma$, and that of one-eighth of the solid, bounded by the planes $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, to have the value $\frac{3}{8}(\alpha+\beta+\gamma)$.

It is worth while to observe, in this place, that from the equation $\rho = q - w$, (where ρ is the generating vector to a family of surfaces, and w is a scalar parameter), which Tait* has assigned as the equation of a *volume*, would be deduced substantially the same results as the foregoing. Let Tq represent the length of the radius vector to the surface which bounds the solid, and let q be written as a function of t, u, θ; then if t, u be taken as the variables which define the surface, Tq and UVq will be functions of t, u exclusively, and θ will turn out to be the angle of q. Thus the equation of the solid may be written

$$q = \phi(t, u, \theta),$$

or

$$Vq = \chi(t, u) \sin \theta .$$

This vector equation will be satisfied for every point within the solid; and it has the same form as the equation $\rho = z\chi(t, u)$, made use of in the foregoing discussion.

*Treatise on Quaternions, p. 61.

APRIL *15th, 1879.*

On the Dynamics of a "Curved Ball."

By Ormond Stone, *Cincinnati, O.*

In his paper "On the Lateral Deviation of Spherical Projectiles,"* Professor Eddy has made a mistake which destroys the force of his argument. In the equation (7)

$$\cos\psi = \sin\delta\sin\theta\cos\phi - \cos\delta\cos\theta,$$

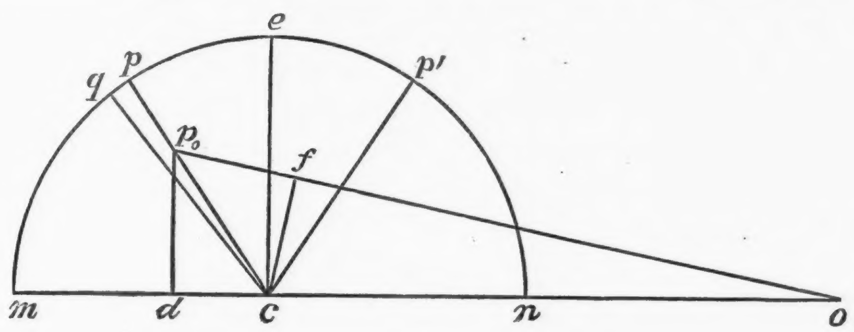
ψ is the arc of a great circle joining dS to m and not that joining it to n'. The cosine of the arc joining dS to n' is

$$\sin\delta\sin\theta\cos\phi + \cos\delta\cos\theta.$$

If a plane be passed through dS, m and o, it is manifest that the pressure dP will lie in this plane, and that the component acting from c toward o will be proportional to the cosine of the angle at c subtended by the arc joining dS to m.

"Again," says Professor Eddy, "it appears from the interpretation given to ψ, . . . that, so long as $\theta < 90°$, more than half the elements dS along this ring between pp' and qq' are within 90° of n', and hence the largest positive value of $\cos\psi$ numerically exceeds its largest negative value." Now, if p be between e and g, all the elements dS along the ring are within 90° of n', although the pressure upon each of these elements in the direction co is negative.

Equation (8) is somewhat complicated, but can be readily integrated for special cases. The following solution of the problem is suggested as preferable. Professor Eddy's nomenclature, etc., are retained as far as practicable.



* This Journal, Vol. II, pp. 85–88.

"In the figure let c be the center of a spherical projectile whose radius is a, and let men be" one-half of "the great circle of the sphere which lies in a horizontal plane. Let us disregard the vertical component of the motion of the projectile; and let c have a horizontal motion of translation, at the instant under consideration, towards e. Also, let the projectile have a motion of rotation about a vertical axis through c in a right-handed direction, *i. e.* from m to e. The motions of translation and rotation, whatever be their relative velocities, can be combined, as is well known, into a single motion of rotation about an instantaneous axis parallel to the vertical axis of rotation through c. This instantaneous axis must intersect the diameter mn, which is perpendicular to the direction of translation ce at some point, as o. Let the instantaneous axis through o be called the axis of z. Also, let the distance oc be designated by the letter b."

Let r be the distance of any element dS of the surface of the sphere from the axis of z. Pass a vertical plane through c, cutting the hemisphere men in a semicircle whose projection is pc, and similarly pass a second plane qc, making an infinitesimal angle $d\mu = pcq$ with pc, and let $\delta = fcp$, $fco = dp_0o = \theta$, $ecp = \mu$, $cp_0 = a \cos \nu$;

$$\therefore \quad cf = a \cos \nu \cos \delta = b \cos \theta ,$$

$$dp_0 = a \cos \nu \cos \mu = r \cos \theta ;$$

$$\therefore \qquad \cos \delta = \frac{b}{r} \cos \mu .$$

Since z is the instantaneous axis of rotation,

$$v = cr ,$$

where v is the velocity of any element of the surface of the sphere and c is a constant.

Let dS be the quadrilateral element of the spherical surface included between the semicircles pc and qc, making an angle $d\mu$ with one another, and two small circles parallel with the horizon having a difference in altitude of $d\nu$ measured in arc on the surface of the sphere. Let p_0 be the projection of dS on the horizontal plane; then dS is ultimately a rectangle the length of whose sides are $ad\nu$ and $a \cos \nu \; d\mu$;

$$\therefore \quad dS = a^2 \cos \nu \; d\mu \; d\nu .$$

If we assume that the pressure dP on dS is toward c and proportional to v^n, where n is a constant > 1, and to $\cos \delta \; dS$ the cross section of the stream of

air which dS meets in its motion, we have

$$dP = c'v^n \cos\delta \, dS = mr^{n-1} \cos\mu \cos\nu \, d\mu \, d\nu,$$

where c' and m are constants and $m = a^2bc^nc'$.

The component dX of dP, acting in the direction co, is $\sin\mu \cos\nu \, dP$;

$$\therefore \; dX = r^{n-1}dQ$$

where $dQ = \frac{1}{2} m \sin 2\mu \cos^2\nu \, d\mu \, d\nu$.

Also, if r' be the distance from the axis of z of an element of the surface dS' having an azimuth $-\mu = ecp'$ and altitude ν, the component of the pressure dP' in the direction co is

$$dX' = -r'^{n-1}dQ;$$

$$\therefore \; d(X + X') = (r^{n-1} - r'^{n-1}) \, dQ.$$

Hence, since r is greater than r', $d(X + X')$ is positive, *i. e.* the deviating pressure acting upon the two elements dS and dS' is from c towards o.

The normal pressures acting upon dS and dS' are evidently greater than the average pressure of the atmosphere. On the other hand, the motion of the ball causes a diminution of pressure upon the elements dS'' and dS''' whose altitude is the same as that of dS and dS', but whose azimuths are $180° - \mu$ and $180° + \mu$. Since the velocity of dS'' is greater than that of dS''', the diminution of the pressure upon dS'' is greater than that upon dS'''. Nevertheless, the increase of pressure in front, caused by an increase in velocity of a body moving through a homogeneous elastic fluid is always greater than the corresponding decrease behind;

$$\therefore \; dP - dP' > dP'' - dP''';$$

consequently, since $d(X + X')$ is positive, $d(X + X' + X'' + X''')$ is also positive; or, in other words, the total lateral pressure upon the four points in question, and hence by integration the total lateral pressure upon the projectile is from c toward o.

Note on Determinants and Duadic Synthemes.

By J. J. Sylvester.

(Continuation. See pp. 89–96 of this Volume.)

The properties of the ω series 1, 1, 2, 8, 50, ... (see p. 94) present some features of interest. These are the numbers of distinct terms in pure skew determinants of the order $2n$ divided by the product of the odd integers inferior to $2n$. Such numbers themselves may be termed the denumerants, and the quotients, when they are so divided, the reduced denumerants of the corresponding determinants; or for greater brevity we may provisionally call these reduced denumerants *skew numbers*. We have found, in what precedes, that

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{1-t}} = \omega_0 + \omega_1 \frac{t}{2} + \omega_2 \frac{t^2}{2.4} + \omega_3 \frac{t^3}{2.4.6} + \ldots$$

From this we may easily obtain

$$\omega_x = \frac{Fx}{2^x},$$

where $Fx = 1 + 1.x + 1.5\frac{x(x-1)}{1.2} + 1.5.9\frac{x(x-1)(x-2)}{1.2.3} + \ldots + \ldots$ $\ldots + 1.5.9\ldots(4x-3)$, which shows that Fx, for all values of x, contains 2^x as a factor, and that if we take x greater than unity, 2^{x+1} will be a factor of Fx. In general, it follows from the fundamental equation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$ that if two consecutive skew numbers ω_c, ω_{c+1} have a common factor, all those of superior orders, and consequently $\frac{Fx}{2^x}$, for all values of x from c upwards, will contain such factor. It becomes then a matter of interest to assign, if possible, a general expression for the greatest common measure of ω_x, ω_{x+1}.

In the first place I say these can have no common odd factor other than unity.

Lemma. It is well known that, in the development of $(1+a)^x$, all the coefficients except the first and last will contain x when it is a prime number. More generally it may easily be shown (and the mode of proof* is too obvious to need setting out) that whatever x may be, any prime number contained in it must either divide any number r, or else the coefficient of a^r in the binomial expression above referred to. Hence we may prove that ω_x and x cannot have a common odd factor other than unity. For if possible, let $x=qp$, where p is a prime number contained in ω_x. Let the qp terms in Fx subsequent to the first term be divided into q groups, each containing p terms. Each of the terms in any one group (except the last) contains a binomial coefficient, which, by virtue of the lemma, will contain p. Moreover, the last term in the kth group will contain the factor $1.5.9\ldots(4kp-1)$.

If p is of the form $4n-3$, the nth term of the series 1, 5, 9, ... will be p, and if it is of the form $4n-1$, the $(3n)$th term will be $3p$; and as $\frac{p+3}{4}$ and $3\,\frac{p+1}{4}$ are each not greater than p (and *à fortiori* not greater than kp) when p is greater than 1, it follows that the last coefficient, as well as all the others in any group, contains p. Hence $Fx=pP+1$, and therefore ω_x, *i. e.* $\frac{Fx}{2^x}$, cannot contain p. Hence the greatest common measure of ω_x and ω_{x+1} is a power of 2.

It will presently be shown by induction (waiting a strict proof)† that $\frac{\omega_{4x-2}}{2^x}$, $\frac{\omega_{4x-1}}{2^x}$, $\frac{\omega_{4x}}{2^x}$, $\frac{\omega_{4x+1}}{2^x}$ are all of them integers, and the first, third and fourth, odd integers; from this it will easily be seen that the greatest common measure of ω_x, ω_{x+1} is $2^{\theta\left(\frac{2x+1}{8}\right)}$, where, in general, $\theta(\mu)$ means the integer

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*Some of the prolixity of the more obvious mode of proof of this lemma may be avoided by the substitution of the following method:

$$\text{Call } (1+t)^n = 1 + A_1t + A_2t^2 + A_2 + A_3t^3 + \ldots, \text{ so that}$$
$$n\,(1+t)^{n-1} = A_1 + 2A_2t + 3A_3t^2 + \ldots$$
$$= B_0 + B_1t + B_2t^2 + \ldots\ldots = \phi t.$$

Suppose $n=qp$: then designating the qth roots of unity by $\rho_1, \rho_2 \ldots \rho_q$, we have

$$\frac{1}{q}\,\Sigma\rho^{q-k}\,\phi(\rho t) = B_kt^k + B_{k+q}t^{k+q} + B_{k+2q}t^{k+2q} + \ldots + B_{k+(p-1)q}t^{k+(p-1)q},$$

and the left hand side of the equation is obviously a multiple of p. Hence, putting t successively equal to 0, 1, 2, 3, ... $(p-1)$, we obtain, by a well-known theorem of determinants,

$$\Delta B_{k+\lambda q} \equiv 0 \text{ [mod } p],$$

where Δ, being the product of the differences of 0, 1, 2, ... $(p-1)$, cannot contain p. Hence $B_{k+\lambda q} \equiv 0$ [mod p], and consequently giving k all values from 0 to $(q-1)$, and λ all values from 0 to $(p-1)$, we see that all the B's, from B_0 to B_{pq-1}, must contain p as a factor as was to be proved.

† Since the above was set up in print, I have found an easy proof, for which see *Postscript*.

nearest to μ. Let us call the above fractions q_{4x-2}, q_{4x-1}, q_{4x}, q_{4x+1}, to which we may give the name of simplified skew numbers. In the subjoined table I have calculated the values of the residues of these numbers by a regular algorithm in respect to *moduli* beginning with 2^{23} and regularly decreasing according to the descending powers of 2. R stands for the words *residue of*.

Modulus.	x	Rq_{4x-2}	Rq_{4x-1}	Rq_{4x}	Rq_{4x+1}
8,388,608	0			1	1
4,194,304	1	1	4	25	209
2,097,152	2	1,087	13,504	194,951	1,088,983
1,048,576	3	929,451	442,068	992,179	576,715
524,288	4	287,913	118,168	393,089	71,201
262,144	5	201,913	14,228	126,417	179,945
131,072	6	51,071	56,656	46,407	127,767
65,536	7	56,531	24,452	15,131	46,739
32,768	8	12,521	29,928	22,753	29,729
16,384	9	14,289	5,412	15,209	14,305
8,192	10	1,119	2,784	4,063	4,751
4,096	11	3,283	3,156	2,331	3,059
2,048	12	1,721	1,632	425	1,801
1,024	13	913	84	1,001	385
512	14	215	240	479	239
256	15	91	132	99	219
128	16	81	8	9	9
64	17	41	36	1	57
32	18	23	0	31	15
16	19	3	4	11	3
8	20	1	0	1	1
4	21	1	0	1	1
2	22	1	0	1	1

From this table it appears that q_{8i-5} is 4 times an odd number, and that q_{8i-1} is 8 times a number which may be odd or even; thus we know the exact

number of times that 2 will divide out all the skew numbers other than those whose orders are of the form $8i-1$, and an inferior limit to that number for that case.

It will further be noticed that, when x is of the form $4i$, or $4i+1$, the simplified skew numbers q_{4x-2}, q_{4x}, q_{4x+1} are all of the form $8\lambda+1$, that when x is of the form $4i+2$ the above named simplified skew numbers are of the form $8\lambda+7$, and when x is of the form $4i+3$, they are of the form $8\lambda+3$.

Before quitting this subject, I think it desirable briefly to refer to other series of integers closely connected with those which I have called *skew numbers.* To this end we may write, in general,

$$e^{\frac{t}{4}}(1-t)^{\frac{4\mu-1}{4}} = 1 + \omega_{1,\mu}\frac{t}{2} + \omega_{2,\mu}\frac{t^2}{2.4} + \omega_{3,\mu}\frac{t^3}{2.4.6} + \ldots,$$

μ being any positive or negative integer, so that $\omega_{x,0}$ is the same as I have called hitherto ω_x. It may then easily be shown that $\omega_{x,\mu+1} = \dfrac{2\omega_{x+1,\mu}-\omega_{x,\mu}}{4\mu+1}$, that $\omega_{x,\mu-1} = \omega_{x,\mu} - 2x\omega_{x-1,\mu}$, and that the equation in differences for $\omega_{x,\mu}$, for μ constant, becomes

$$\omega_{x,\mu} = (2x+2\mu-1)\,\omega_{x-1,\mu} - (x-1)\,\omega_{x-2,\mu},$$

with the initial conditions $\omega_{0,\mu}=1$, $\omega_{1,\mu}=2\mu+1$. Also, it is clear from the definition, that the explicit value of $\omega_{x,\mu}$ in a series becomes

$$\frac{1}{2^x}\left\{1 + (4\mu+1)\,x + (4\mu+1)(4\mu+5)\,x\frac{x-1}{2} \right.$$
$$\left. + (4\mu+1)(4\mu+5)(4\mu+9)\,x\frac{x-1}{2}\cdot\frac{x-2}{3} + \ldots\right\},$$

which is easily seen to verify the equation

$$2\omega_{x,\mu} - \omega_{x-1,\mu} = (4\mu+1)\,\omega_{x-1,\mu+1}.^*$$

We might call the $\omega_{x,\mu}$ series skew numbers of the μth degree, and, as for the case of $\mu=0$, so it may be shown in general that two consecutive skew numbers of the same degree can have no common odd factor. Also, it remains true that the greatest common factor of any two consecutive skew numbers of the same degree and the orders x, $x+1$, is $2^{\theta\left(\frac{2x+1}{8}\right)}$; $\omega_{4x-2,\mu}$, $\omega_{4x-1,\mu}$, $\omega_{4x,\mu}$, $\omega_{4x+1,\mu}$ being all divisible by 2^x, and the resulting quotients being, the first, third

*And of course, in general, the equation

$$\lambda u_{x,y} - u_{x-1,y} + \phi y u_{x-1,y+\delta} = 0,$$

with the condition that $u_{0,y}$ is constant, has for its integral

$$u_{x,y} = \frac{c}{\lambda^x}\left\{1 - \phi y x + \phi y \phi(y+\delta)\,x\frac{x-1}{2} - \phi y\phi(y+\delta)\,\phi(y+2\delta)\,x\frac{x-1}{2}\,\frac{x-2}{3} + \ldots\right\}.$$

and fourth of them, always odd integers, and the second divisible by 4 or some higher power of 2 when μ is even, but only by the first power of 2 when μ is odd. But it would carry me too far away from the original object of this note, and from other investigations of more pressing moment to myself, to pursue further the theory of general skew numbers, which, however, seems to me to be well worthy of the study of arithmeticians.

I will only stop to point out that the rule for the greatest common measure of ω_x and ω_{x+1}, serves to prove the rule for the general case of $\omega_{x,\mu}$ and $\omega_{x+1,\mu}$. Thus suppose μ to be positive. Then since $\omega_{k,1} = 2\omega_{k+1} - \omega_k$, and $\omega_{4k-2} = 2^k(2\lambda + 1)$, $\omega_{4k-1} = 2^{k+1}\tau$, $\omega_{4k} = 2^k(2\nu + 1)$, $\omega_{4k+1} = 2^k(2\pi + 1)$, $\omega_{4k+2} = 2^{k+1}(2\rho + 1)$; it follows that
$\omega_{4k-2,1} = 2^k(2\lambda' + 1)$, $\omega_{4k-1,1} = 2^{k+1}\tau'$, $\omega_{4k,1} = 2^k(2\nu' + 1)$, and $\omega_{4k+1,1} = 2^k(2\pi' + 1)$.

It is obvious further that, τ being even, τ' is odd. So again from these results we may, in like manner, deduce $\omega_{4k-2,2} = 2^k(2\lambda'' + 1)$, $\omega_{4k-1,2} = 2^{k+1}\tau''$, $\omega_{4k,2} = 2^k(2\nu'' + 1)$, $\omega_{4k+1,2} = 2^k(2\pi'' + 1)$, subject also to the remark that, τ' being odd, τ'' is even, and so on continually, τ being alternately even and odd. Again if μ is negative, we may, in like manner, by means of the formula $\omega_{k,\mu-1} = \omega_{k,\mu} - 2k\omega_{k-1,\mu}$, pass successively from the case of ω_k to that of $\omega_{k,-1}: \omega_{k,-2}: \ldots \omega_{k,-\mu}$, and establish precisely the same conclusion in regard to powers of 2 as for the case of μ positive, and it will be remembered that I have already shown how to establish that $\omega_{k,\mu}$ and $\omega_{k+1,\mu}$ have no common *odd* factor.

In the first note on this subject (Vol. II, No. 1, of the *Journal*) I showed how a general determinant could be completely represented by means of systems of cycles and that accordingly the terms in the total development would split up into families, as many in number as there are indefinite partitions of the index of the order of the determinant—the particular mode of aggregation depending upon the term chosen to represent the product of the elements in the principal diagonal, so that for the order n there would be $1.2.3\ldots n$ distinct modes of distribution into families. This gives rise to a theory of transformation of cycles, corresponding to a transposition of the rows or columns of the matrix. Thus *ex. gr.* suppose the *umbrae* to be 1, 2, 3, ... n: r, s signifying the element in the rth row and sth column. Then if we interchange the mth and nth columns, this will have the effect of changing pm into pn and pn into pm.

Suppose now that a term of the developed determinant is expressed by a system of cycles such that m and n lie in two distinct cycles, say Xm and nY, where X, Y are each of them single elements, or aggregates of single elements; then the effect of the interchange will be to bring these cycles into the single cycle $XnYm$. If Xm, nY were both odd ordered or both even ordered cycles, their sum will be even ordered, and the number of *even* cycles will be increased or diminished by unity; so if one was of odd and the other of even order, their sum will be of odd order, and the number of even cycles will be diminished by unity. In either case, therefore, the sign, which depends on the *parity* of the number of even cycles, is reversed.

Again, suppose m and n to lie in the same cycle $mXnY$. Then the effect of the interchange will be to break this up into two cycles mX, nY, and for the same reason as above the sign will be reversed. Thus the sign of every term in the development will, we see, be reversed, as we know *à priori* ought to be the case.

I shall conclude with applying the formula $\omega_x = \frac{Fx}{2^x}$ to determining the *asymptotic* mean value of the coefficients in a skew determinant of the order $2x$, *i. e.* the function of x to which the mean value of the coefficients converges when x is taken indefinitely great. We know that all the coefficients, both in this case and in that of a symmetrical determinant, are different powers of 2; to find the mean of the indices of these powers would be seemingly an investigation of considerable difficulty, but there will be little or none in finding the ultimate expression for the mean of the coefficients themselves, or, which is the same thing, the first term in the function which expresses this mean in terms of descending powers of x. We shall find that, for symmetrical determinants, this is a certain multiple of the square root and, for skew determinants, of the fourth root of x, as I proceed to show.

From the equation

$$2^x\omega_x = 1 + x + 5x\,\frac{x-1}{2} + \ldots + (1.5\ldots(4x-3)),$$

we have, when $x=\infty$,

$$2^x\omega_x = 1.5.9\ldots\overline{4x-3}\left\{1+\frac{x}{4x-3}+\frac{1}{2}\,\frac{x(x-1)}{(4x-3)(4x-7)}+\ldots\right\}$$

$$= e^{\frac{1}{4}}.1.5.9\ldots\overline{4x-3}.$$

The number of terms in the Pfaffian (the square root of the determinant taken with suitable algebraical sign) being $1.3.5\ldots\overline{2x-1}$ and—as follows from

what was shown in the first note—cancelling being out of the question, the sum of the coefficients all taken positively in the determinant itself will be $(1.3.5\ldots\overline{2x-1})^2$. Hence the mean value required is $(1.3.5\ldots\overline{2x-1})^2$ divided by $1.3.5\ldots\overline{2x-1}\,\omega_x$, to express which quotient in exact terms we may make use of the formula

$$\frac{a(a+\delta)(a+2\delta)\ldots(a+x\delta)}{b(b+\delta)(b+2\delta)\ldots(b+x\delta)}=\frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}}x^{\frac{a-b}{\delta}}.$$

For the mean value is

$$\frac{1}{e^{\frac{1}{4}}}\cdot\frac{1.3.5\ldots(2x-1)}{2.4.6\ldots(2x)}\cdot\frac{4.8.12\ldots(4x)}{1.5.9\ldots(4x-3)}=\frac{1}{e^{\frac{1}{4}}}\cdot\frac{1}{\Gamma\frac{1}{2}}x^{-\frac{1}{2}}.\Gamma\frac{1}{4}x^{\frac{3}{4}}=\frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{\pi}}x^{\frac{1}{4}}.$$

If we write this under the form $Qx^{\frac{1}{4}}$, we have

$$Q=\frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\Gamma\frac{1}{2}},$$

$$\begin{aligned}\log Q&=\log\Gamma\frac{5}{4}+\log 2-\log\Gamma\frac{3}{2}-\frac{1}{4}\log e\\&=9.9573211+.3010300-9.9475449-.1085736\\&=.2022326,\\ \text{or } Q&=1.59306.\end{aligned}$$

This result as may easily be seen remains unaffected when, instead of a pure skew determinant, one is taken in which the diagonal terms retain general values. The effect of this change will be to increase the numerator and denominator of the fraction which expresses the mean value, in the proportion of $\frac{e^2+1}{2e}$ to 1.

Finally, as regards the ultimate mean value of the coefficients of symmetrical determinants. This, for one of the order x, by virtue of Professor Cayley's formula previously given, will be the reciprocal of the coefficient of t^x in $\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}$. It may readily be shown in general that, ϕt being any series of integer powers of t, the coefficient of t^x (when x becomes infinite) in $\frac{e^{\phi t}}{\sqrt{1-t}}$

is in a ratio of equality to the coefficient of t^x in $\dfrac{e^{(\phi 1)t}}{\sqrt{1-t}}$, so that in the present case this coefficient is the same as the coefficient of t^x in $\dfrac{e^{\frac{3}{4}t}}{\sqrt{1-t}}$, *i. e.* in

$$\left(1+\frac{1}{2}t+\frac{1.3}{2.4}t^2+\ldots+\frac{1.3.5\ldots(2x-1)}{2.4.6\ldots 2x}t^x+\ldots\right)$$
$$\times\left(1+\frac{3}{4}t+\left(\frac{3}{4}\right)^2\frac{t^2}{2}+\ldots+\left(\frac{3}{4}\right)^x\frac{t^x}{1.2\ldots x}\right),$$

which is obviously, when x is infinite, equal to $\dfrac{1.3.5\ldots(2x-1)}{2.4.6\ldots 2x}e^{\frac{3}{4}}$. Hence the ultimate mean value of the coefficients is $\dfrac{1}{e^{\frac{3}{4}}}\dfrac{2.4.6\ldots 2x}{1.3.5\ldots(2x-1)}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$.

For a symmetrical determinant in which all the diagonal terms are wanting, the numerator of the fraction giving the mean value becomes $e^{-1}(1.2.3\ldots x)$ and the denominator is $(1.2.3\ldots x)$ into the coefficient of t^x in $\dfrac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}$, which is the same as in $\dfrac{e^{-\frac{1}{4}t}}{\sqrt{1-t}}$. The result then is $\dfrac{\pi^{\frac{1}{2}}e^{\frac{1}{4}}}{e}\sqrt{x}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$ as before. It may perhaps be just worth while to notice that the *skew numbers* (the ω's of the text) may be put under the form of a determinant, the nature of which is sufficiently indicated by the annexed diagram.

1	1	0	0	0	0	0
1	3	2	0	0	0	0
0	1	5	3	0	0	0
0	0	1	7	4	0	0
0	0	0	1	9	5	0
0	0	0	0	1	11	6
0	0	0	0	0	1	13

The successive principal minors in this matrix represent the successive skew numbers of all orders from 1 to 6 inclusive.

Postscript.

Since $\omega_{x+1} = (2x+1)\,\omega_x - x\omega_{x-1}$, we have
$\omega_{x+2} = (4x^2 + 7x + 2)\,\omega_x - (2x^2 + 3x)\,\omega_{x-1}$,
$\omega_{x+3} = (8x^3 + 32x^2 + 34x + 8)\,\omega_x - (4x^3 + 15x^2 + 13x)\,\omega_{x-1}$,
$\omega_{x+4} = (16x^4 + 116x^3 + 273x^2 + 231x + 50)\,\omega_x - (8x^3 + 56x^3 + 122x^2 + 82x)\,\omega_{x-1}$.
Suppose now that, for a given value of i, $q_{4i-2} = \frac{\omega_{4i-2}}{2^i} = 2\lambda + 1$, $q_{4i-1} = \frac{\omega_{4i-1}}{2^i} = 4\mu$, $q_{4i} = \frac{\omega_{4i}}{2^i} = 2\nu + 1$ and $q_{4i+1} = \frac{\omega_{4i+1}}{2^i} = 2\rho + 1$. Call $\omega_{x+4} = E_x\omega_x - F_x\omega_{x-1}$. Then when $x \equiv \pm 2$, $F_x \equiv 4$ [mod 8], and therefore, assuming that $q_{4i-3} = \frac{\omega^{4i-3}}{2^{i-1}}$ is odd, $\frac{F_{4i-2}\,\omega_{4i-3}}{2^{i+1}}$ is odd. Also, $E_{4i-2} \equiv 462 + 50 \equiv 0$ [mod 4], and consequently $\frac{E_{4i-2}\omega_{4i-2}}{2^{i+1}}$ is even; hence $q_{4i+2} = \frac{\omega_{4i+2}}{2^{i+1}}$ is integer and odd. Again when $x = 4i - 1$, $E_x \equiv 1 - 3 + 50 \equiv 0$ [mod 4], and $F_x \equiv 122 - 82 \equiv 0$ [mod 8]; hence $q_{4i+3} = \frac{\omega_{4i+3}}{2^{i+1}}$ is an integer divisible by 4. Again, when $x = 4i$, $E_{4i} \equiv 2$ and $F_{4i} \equiv 0$ [mod 4]; hence $q_{4i+4} = \frac{\omega_{4i+4}}{2^{i+1}}$ is integer and odd; and when $x = 4i + 1$, $E_{4i+1} \equiv 2$ and $F_{4i+1} \equiv 0$ [mod 4]; hence $q_{4i+5} = \frac{\omega_{4i+5}}{2^{i+1}}$ is integer and odd.

Thus it has been shown that if it be true up to $\lambda = i$ that $\frac{\omega_{4\lambda-2}}{2^\lambda}$, $\frac{\omega_{4\lambda-1}}{2^{\lambda+2}}$, $\frac{\omega_{4\lambda}}{2^\lambda}$, $\frac{\omega_{\lambda 4+1}}{2^\lambda}$ are all integer, and the first, third and fourth odd integers, the same proposition can be affirmed for all superior values of i, and being true for ω_0, ω_1, ω_2, ω_3, the quotients corresponding to which are 1, 1, 1, 1, the theorem is true universally. It is inconceivable that it could have occurred to any human being to lay down so singular a train of induction as the one above employed, unless previously prompted to do so by an *à priori* perception of the law to be established, acquired through a preliminary study and direct inspection of the earlier terms in the series of numbers to which it applies. Here then we have a salient example (if any were needed) of the importance of the part played by the *faculty of observation* in the discovery and establishment of pure mathematical laws.

Tables of the Generating Functions and Groundforms for the Binary Quantics of the First Ten Orders.

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In what follows, "G. F." stands for the words *Generating Function.* In the Generating Functions, the exponents of the letter a refer to degree in the coefficients, and the exponents of the letter x to order in the variables. The Generating Functions for differentiants take account only of degree in the coefficients, without regard to the order in the variables of the covariant of which the differentiant is the "*source.*" In the *tabulated* numerators of the Generating Functions, the *minus* sign is placed *over* instead of *to the left of* the number which it affects.

QUADRIC.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-a^2)}$.

G. F. for covariants, $\dfrac{1}{(1-a^2)(1-ax^2)}$.

Groundforms: 1 of deg. 1, ord. 2; 1 of deg. 2, ord. 0.

CUBIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)(1-a^4)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax+a^2x^2}{(1-a^4)(1-ax)(1-ax^3)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^3}{(1-a^4)(1-a^2x^2)(1-ax^3)}$.

Groundforms: 1 of deg. 1, ord. 3; 1 of deg. 2, ord. 2; 1 of deg. 3, ord. 3; 1 of deg. 4, ord. 0.

QUARTIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)^2(1-a^3)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax^2+a^2x^4}{(1-a^2)(1-a^3)(1-ax^2)(1-ax^4)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^6}{(1-a^2)(1-a^3)(1-a^2x^4)(1-ax^4)}$.

Groundforms: 1 of deg. 1, ord. 4; 1 of deg. 2, ord. 0; 1 of deg. 2, ord. 4; 1 of deg. 3, ord. 0; 1 of deg. 3, ord. 6.

QUINTIC.

G. F. for differentiants,

$$\frac{1+a^2+3a^3+3a^4+5a^5+4a^6+6a^7+6a^8+4a^9+5a^{10}+3a^{11}+3a^{12}+a^{13}+a^{15}}{(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5)$.

Numerator: $1+a(-x-x^3)+a^2(x^2+x^4+x^6)-a^3x^7+a^4x^4+a^5(x+x^3-x^5)$
$+a^6(-1-x^4)+a^7(2x+x^3+x^5)+a^8(-x^2-x^4-2x^6)$
$+a^9(x^3+x^7)+a^{10}(x^2-x^4-x^6)-a^{11}x^3+a^{12}+a^{13}(-x-x^3-x^5)$
$+a^{14}(x^4+x^6)-a^{15}x^7$.

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^2x^6)(1-ax^5)$.

Numerator: $1+a^3(x^3+x^5+x^9)+a^4(x^4+x^6)+a^5(x+x^3+x^7-x^{11})$
$+a^6(x^2+x^4)+a^7(x+x^5-x^9)+a^8(x^2+x^4)+a^9(x^3+x^5-x^7)$
$+a^{10}(x^2+x^4-x^{10})+a^{11}(x+x^3-x^9)+a^{12}(x^2-x^8-x^{10})$
$+a^{13}(x-x^7-x^9)+a^{14}(x^4-x^6-x^8)+a^{15}(-x^7-x^9)$
$+a^{16}(x^2-x^6-x^{10})+a^{17}(-x^7-x^9)+a^{18}(1-x^4-x^8-x^{10})$
$+a^{19}(-x^5-x^7)+a^{20}(-x^2-x^6-x^8)-a^{23}x^{11}$.

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	1	2	3	4	5	6	7	9
1						1			
2			1				1		
3				1		1			1
4	1				1		1		
5		1		1				1	
6			1		1				
7		1				1			
8	1		1						
9				1					
11		1							
12	1								
13		1							
18	1								

SEXTIC.

G. F. for differentiants, $\dfrac{1+a^2+3a^3+4a^4+4a^5+4a^6+3a^7+a^8+a^{10}}{(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-ax^2)(1-ax^4)(1-ax^6)$.

Numerator: $1+a(-x^2-x^4)+a^2(-1+x^4+x^6+x^8)+a^3(-1+2x^2+x^4-x^{10})$
$+a^4(x^2-x^6-x^8)+a^5(-x^6-x^8+x^{10})+a^6(1-x^2-x^8+x^{10})$
$+a^7(1-x^2-x^4)+a^8(-x^2-x^4+x^8)+a^9(-1+x^6+2x^8-x^{10})$
$+a^{10}(x^2+x^4+x^6-x^{10})+a^{11}(-x^6-x^8)+a^{12}x^{10}$.

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)(1-a^{10})(1-a^2x^4)(1-a^2x^8)(1-ax^6)$.

Numerator: $1+a^3(x^2+x^6+x^8+x^{12})+a^4(x^4+x^6+x^{10})+a^5(x^2+x^4+x^8-x^{16})$
$+a^6(x^4+2x^6)+a^7(x^2+x^4+x^8-x^{12})+a^8(x^2+x^4+x^6-x^{14})$
$+a^9(x^4+x^6-x^{10}-x^{12})+a^{10}(x^2+x^4-x^{12}-x^{14})+a^{11}(x^4$
$+x^6-x^{10}-x^{12})+a^{12}(x^2-x^{10}-x^{12}-x^{14})+a^{13}(x^4-x^8-x^{12}-x^{14})$
$+a^{14}(-2x^{10}-x^{12})+a^{15}(1-x^8-x^{12}-x^{14})+a^{16}(-x^6-x^{10}-x^{12})$
$+a^{17}(-x^4-x^8-x^{10}-x^{14})-a^{20}x^{16}$.

Table of Groundforms.

DEGREE IN THE COEFFICIENTS.	ORDER IN THE VARIABLES. 0	2	4	6	8	10	12
1				1			
2	1		1		1		
3		1		1	1		1
4	1		1	1		1	
5		1	1		1		
6	1			2			
7		1	1				
8		1					
9			1				
10	1	1					
12		1					
15	1						

* This is not strictly the minimum form, its numerator and denominator being divisible by $1-a$; it is, however, the lowest form to which the fraction can be reduced when the factors of the denominator are all of the forms $1-a^r$, $1-a^rx^s$. The same remark applies to the "reduced form" in the case of the decimic.

SEPTIMIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$.

Numerator: $1+2a^2+6a^3+10a^4+19a^5+28a^6+44a^7+61a^8+79a^9$
$+102a^{10}+129a^{11}+156a^{12}+173a^{13}+196a^{14}+215a^{15}$
$+230a^{16}+231a^{17}+231a^{18}+230a^{19}+215a^{20}+196a^{21}$
$+173a^{22}+156a^{23}+129a^{24}+102a^{25}+79a^{26}+61a^{27}+44a^{28}$
$+28a^{29}+19a^{30}+10a^{31}+6a^{32}+2a^{33}+a^{35}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-ax)(1-ax^3)$
$(1-ax^5)(1-ax^7)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^0	1														
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$									
a^2			1		1		2		1		1				
a^3								$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^4					2				1						1
a^5		1		2						$\bar{1}$		$\bar{1}$			
a^6	$\bar{1}$		2		$\bar{1}$				$\bar{1}$		$\bar{1}$		1		
a^7		4		1		3				$\bar{1}$		1			
a^8	2		$\bar{1}$				$\bar{3}$		$\bar{3}$		$\bar{1}$		$\bar{1}$		
a^9		1		3		1		$\bar{1}$		2				2	
a^{10}	$\bar{1}$		4				$\bar{1}$		$\bar{2}$		$\bar{2}$				$\bar{1}$
a^{11}		5		3		2		$\bar{1}$		$\bar{2}$		$\bar{1}$		1	
a^{12}	5		1				$\bar{4}$		$\bar{6}$		$\bar{4}$		$\bar{1}$		2
a^{13}		1				$\bar{4}$		$\bar{4}$		$\bar{1}$		1		4	
a^{14}	2		5		1		1		$\bar{2}$				3		$\bar{1}$
a^{15}		8		$\bar{1}$		$\bar{1}$		$\bar{7}$		$\bar{5}$		$\bar{1}$		$\bar{1}$	
a^{16}	6		3		3		$\bar{4}$		$\bar{3}$				$\bar{1}$		5
a^{17}		$\bar{1}$		$\bar{2}$		$\bar{9}$		$\bar{8}$		$\bar{4}$		$\bar{3}$		4	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^{18}	2		6		1		2		2		1		6		2
a^{19}		4		$\bar{3}$		$\bar{4}$		$\bar{8}$		$\bar{9}$		$\bar{2}$		$\bar{1}$	
a^{20}	5		$\bar{1}$				$\bar{3}$		$\bar{4}$		3		3		6
a^{21}		$\bar{1}$		$\bar{1}$		$\bar{5}$		$\bar{7}$		$\bar{1}$		$\bar{1}$		3	
a^{22}	$\bar{1}$		3				$\bar{2}$		1		1		5		2
a^{23}		4		1		$\bar{1}$		$\bar{4}$		$\bar{4}$				1	
a^{24}	2		$\bar{1}$		$\bar{4}$		$\bar{6}$		$\bar{4}$				1		5
a^{25}		1		$\bar{1}$		$\bar{2}$		$\bar{1}$		2		3		5	
a^{26}	$\bar{1}$				$\bar{2}$		$\bar{2}$		$\bar{1}$				4		$\bar{1}$
a^{27}		2				2		$\bar{1}$		1		3		1	
a^{28}			$\bar{1}$		$\bar{1}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		2
a^{29}				1		$\bar{1}$				3		1		4	
a^{30}			1		$\bar{1}$		$\bar{1}$				$\bar{1}$		2		$\bar{1}$
a^{31}				$\bar{1}$		$\bar{1}$						2		1	
a^{32}	1						1				2				
a^{33}		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$							
a^{34}					1		1		2		1		1		
a^{35}										$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^{36}															1

Owing to the non-existence of an irreducible invariant whose degree is 10, or any multiple of 10, no representative generating function with a *finite* numerator can be obtained for the septimic; the factor $1-a^{10}$ in the denominator has to be got rid of by dividing numerator and denominator by it, or, in other words, by striking it out of the denominator and multiplying the numerator by the infinite series $1+a^{10}+a^{20}+\ldots.$ We thus obtain:

G. F. for covariants, representative form, (with infinite numerator),

Denominator: $(1-a^4)(1-a^8)(1-a^{12})^2(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$.

Numerator: (Given to the terms containing the 45th power of a, inclusive; after which, each column can be continued by repeating *the last five coefficients* occurring in it, *ad inf.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^3				1		1		1		1		1				1								
a^4					2		1		2		1				1									
a^5		1		2		2		2		2								$\bar{1}$				$\bar{1}$		
a^6			3		2		3		3				2		$\bar{1}$		$\bar{1}$							
a^7		3		2		4		4				1				$\bar{2}$				$\bar{1}$				1
a^8	2		3		4		6		1		3		$\bar{1}$		$\bar{2}$				$\bar{1}$					
a^9		3		5		7		1		4				$\bar{2}$		$\bar{1}$		$\bar{2}$				1		
a^{10}			5		8		6		4		1		$\bar{4}$				$\bar{3}$		$\bar{1}$					
a^{11}		5		8		8		8		4		$\bar{4}$		$\bar{1}$		$\bar{5}$		$\bar{1}$						
a^{12}	4		9		9		12		4		$\bar{1}$		$\bar{3}$		$\bar{5}$		$\bar{6}$				$\bar{1}$		1	
a^{13}		9		9		12		6		$\bar{1}$		$\bar{3}$		$\bar{8}$		$\bar{9}$		$\bar{3}$		$\bar{1}$		1		
a^{14}	4		9		13		11		$\bar{1}$		$\bar{3}$		$\bar{9}$		$\bar{10}$		$\bar{7}$		$\bar{2}$				3	
a^{15}		9		12		16		3		2		$\bar{10}$		$\bar{11}$		$\bar{8}$		$\bar{3}$				3		2
a^{16}	5		14		15		12		1		$\bar{5}$		$\bar{16}$		$\bar{9}$		$\bar{9}$		$\bar{1}$		3		3	
a^{17}		12		15		16		6		$\bar{3}$		$\bar{17}$		$\bar{13}$		$\bar{15}$		$\bar{5}$		2		3		
a^{18}	9		14		15		14		$\bar{3}$		$\bar{13}$		$\bar{20}$		$\bar{15}$		$\bar{15}$		2		2		5	
a^{19}		15		16		18				$\bar{8}$		$\bar{18}$		$\bar{20}$		$\bar{19}$		$\bar{3}$		3		5		4
a^{20}	7		14		18		12		$\bar{10}$		$\bar{16}$		$\bar{25}$		$\bar{19}$		$\bar{12}$		2		5		9	
a^{21}		14		17		19		$\bar{1}$		$\bar{8}$		$\bar{27}$		$\bar{25}$		$\bar{16}$		$\bar{2}$		4		8		4
a^{22}	9		17		19		11		$\bar{8}$		$\bar{18}$		$\bar{31}$		$\bar{17}$		$\bar{15}$		6		9		9	
a^{23}		17		19		18		$\bar{3}$		$\bar{13}$		$\bar{31}$		$\bar{25}$		$\bar{21}$		$\bar{4}$		9		9		5
a^{24}	8		17		17		10		$\bar{12}$		$\bar{27}$		$\bar{32}$		$\bar{22}$		$\bar{16}$		9		9		12	
a^{25}		18		17		19		$\bar{6}$		$\bar{17}$		$\bar{31}$		$\bar{28}$		$\bar{22}$		3		10		12		9
a^{26}	9		18		18		11		$\bar{17}$		$\bar{23}$		$\bar{34}$		$\bar{21}$		$\bar{10}$		10		14		15	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{27}		17		17		19		$\overline{9}$		$\overline{16}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		3		13		14		7
a^{28}	8		17		18		9		$\overline{16}$		$\overline{26}$		$\overline{38}$		$\overline{18}$		$\overline{13}$		14		15		14	
a^{29}		18		19		17		$\overline{8}$		$\overline{16}$		$\overline{36}$		$\overline{25}$		$\overline{21}$		6		16		16		9
a^{30}	9		18		18		10		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{19}$		$\overline{11}$		16		15		17	
a^{31}		17		17		17		$\overline{8}$		$\overline{19}$		$\overline{36}$		$\overline{29}$		$\overline{19}$		8		15		17		8
a^{32}	9		18		18		8		$\overline{18}$		$\overline{26}$		$\overline{35}$		$\overline{19}$		$\overline{10}$		17		17		18	
a^{33}		18		18		18		$\overline{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		18		9
a^{34}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\overline{8}$		17		17		17	
a^{35}		18		17		18		$\overline{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		8
a^{36}	9		19		18		9		$\overline{18}$		$\overline{25}$		$\overline{34}$		$\overline{17}$		$\overline{9}$		17		19		18	
a^{37}		17		17		18		$\overline{9}$		$\overline{18}$		$\overline{37}$		$\overline{26}$		$\overline{18}$		9		17		17		9
a^{38}	9		17		17		9		$\overline{18}$		$\overline{26}$		$\overline{37}$		$\overline{18}$		$\overline{9}$		18		17		17	
a^{39}		18		19		17		$\overline{9}$		$\overline{17}$		$\overline{34}$		$\overline{25}$		$\overline{18}$		9		18		19		9
a^{40}	9		17		18		9		$\overline{18}$		$\overline{27}$		$\overline{35}$		$\overline{17}$		$\overline{9}$		18		17		18	
a^{41}		17		17		17		$\overline{8}$		$\overline{18}$		$\overline{36}$		$\overline{28}$		$\overline{17}$		9		17		17		8
a^{42}	9		18		18		8		$\overline{17}$		$\overline{26}$		$\overline{34}$		$\overline{18}$		$\overline{9}$		18		18		18	
a^{43}		18		18		18		$\overline{9}$		$\overline{18}$		$\overline{34}$		$\overline{26}$		$\overline{17}$		8		18		18		9
a^{44}	8		17		17		9		$\overline{17}$		$\overline{28}$		$\overline{36}$		$\overline{18}$		$\overline{8}$		17		17		17	
a^{45}		18		17		18		$\overline{9}$		$\overline{17}$		$\overline{35}$		$\overline{27}$		$\overline{18}$		9		18		17		9

etc. etc. etc.

Table of Groundforms.

Degree in the Coefficients. / Order in the Variables.	0	1	2	3	4	5	6	7	8	9	10	11	14	15
1								1						
2			1				1				1			
3				1		1		1		1		1		1
4	1				2		1		2		1		1	
5		1		2		2		2		2				
6			3		2		2		2					
7		3		2		4		2						
8	3		3		3		3							
9		3		5		2								
10			4		3									
11		5		3										
12	6		6											
13		7												
14	4													
15		3												
16	2													
17		2												
18	9													
22	1													

OCTAVIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)^2(1-a^4)(1-a^5)(1-a^7)$.

Numerator: $1+2a^2+6a^3+12a^4+19a^5+25a^6+31a^7+36a^8+38a^9+36a^{10}$
$+31a^{11}+25a^{12}+19a^{13}+12a^{14}+6a^{15}+2a^{16}+a^{18}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}
a^0	1									
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$						
a^2			1	1	2	1	1			
a^3			1			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^4			2							1
a^5		1	2		$\bar{1}$		$\bar{1}$			
a^6		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^7		2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^8	1	2			$\bar{2}$	$\bar{2}$	$\bar{2}$	1		
a^9	1	2	$\bar{2}$		$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	
a^{10}	1	1	$\bar{2}$		$\bar{2}$	$\bar{1}$			1	
a^{11}		1	$\bar{1}$		$\bar{1}$	$\bar{1}$		$\bar{1}$	1	
a^{12}		1			$\bar{1}$	$\bar{2}$		$\bar{2}$	1	1
a^{13}		1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{2}$	2	1
a^{14}			1	$\bar{2}$	$\bar{2}$	$\bar{2}$			2	1
a^{15}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1	2	
a^{16}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1	
a^{17}				$\bar{1}$		$\bar{1}$		2	1	
a^{18}	1							2		
a^{19}		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$			1		
a^{20}				1	1	2	1	1		
a^{21}							$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{22}										1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^2x^4)$
$(1-a^2x^8)(1-a^2x^{12})(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}
a^0	1															
a^3			1	1	1	1	1	1		1						
a^4			2	1	1	2	1	1		1						
a^5		1	2	2	1	3		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		
a^6		1	2	3	2	2		1		$\bar{1}$		$\bar{1}$				
a^7		2	2	3	2	2		$\bar{1}$		$\bar{2}$		$\bar{1}$				1
a^8	1	2	2	3	3	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$					
a^9	1	3	1	3	2	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{1}$		1		
a^{10}	1	2	1	2	1	$\bar{2}$	$\bar{2}$	$\bar{5}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{1}$		1		
a^{11}		2	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{4}$	$\bar{4}$	$\bar{3}$			2		
a^{12}		1	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{3}$			2	1	
a^{13}		1	2			$\bar{3}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	1	
a^{14}			2			$\bar{3}$	$\bar{4}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	2	
a^{15}			1		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{2}$	1	2	1	2	1
a^{16}			1		$\bar{1}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	2	3	1	3	1
a^{17}						$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	3	3	2	2	1
a^{18}	1				$\bar{1}$		$\bar{2}$		$\bar{1}$		2	2	3	2	2	
a^{19}					$\bar{1}$		$\bar{1}$		1		2	2	3	2	1	
a^{20}			$\bar{1}$		$\bar{1}$		$\bar{1}$		1		3	1	2	2	1	
a^{21}							1		1	1	2	1	1	2		
a^{22}							1		1	1	1	1	1	1		
a^{25}																1

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	18
1					1				
2	1		1		1		1		
3	1		1	1	1	1	1	1	1
4	1		2	1	1	2	1	1	1
5	1	1	2	2	1	3		1	
6	1	1	2	3	1	1			
7	1	2	2	3					
8	1	2	2	2					
9	1	3	1						
10	1	2							
11		2							
12		1							

Nonic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$
$(1-a^{14})(1-a^{16})$.

Numerator: $1+3a^2+10a^3+23a^4+49a^5+93a^6+172a^7+289a^8+457a^9$
$+701a^{10}+1036a^{11}+1477a^{12}+2023a^{13}+2720a^{14}+3568a^{15}$
$+4573a^{16}+5702a^{17}+7013a^{18}+8466a^{19}+10043a^{20}+11672a^{21}$
$+13400a^{22}+15155a^{23}+16880a^{24}+18487a^{25}+20013a^{26}$
$+21392a^{27}+22539a^{28}+23398a^{29}+24013a^{30}+24355a^{31}$
$+24355a^{32}+24013a^{33}+23398a^{34}+22539a^{35}+21392a^{36}$
$+20013a^{37}+18487a^{38}+16880a^{39}+15155a^{40}+13400a^{41}$
$+11672a^{42}+10043a^{43}+8466a^{44}+7013a^{45}+5702a^{46}+4573a^{47}$
$+3568a^{48}+2720a^{49}+2023a^{50}+1477a^{51}+1036a^{52}+701a^{53}$
$+457a^{54}+289a^{55}+172a^{56}+93a^{57}+49a^{58}+23a^{59}+10a^{60}$
$+3a^{61}+a^{63}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-a^{14})(1-a^{16})$
$(1-ax)(1-ax^3)(1-ax^5)(1-ax^7)(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$																
a^2			1		1		2		2		2		1		1									
a^3								$\bar{1}$		$\bar{1}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$		$\bar{1}$				
a^4	1				2		1		2				1		1		1		1		1		1	
a^5				2		1				$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$		$\bar{1}$						$\bar{1}$
a^6	$\bar{1}$		4		1		3										1		1		1			
a^7		5		5		5		1		$\bar{1}$		$\bar{3}$		$\bar{2}$		$\bar{2}$						$\bar{1}$		
a^8	5		3		4		$\bar{2}$		$\bar{3}$		$\bar{7}$		$\bar{5}$		$\bar{3}$		1		2				1	
a^9		5		8		2		1		$\bar{4}$		$\bar{2}$		$\bar{1}$		3		3		2		1		$\bar{1}$
a^{10}	3		15		5		5		$\bar{3}$		$\bar{7}$		$\bar{7}$		$\bar{2}$		1		$\bar{1}$		1		$\bar{2}$	
a^{11}		17		11		9		$\bar{2}$		$\bar{10}$		$\bar{16}$		$\bar{5}$		$\bar{3}$		2		4		$\bar{1}$		1
a^{12}	18		14		15		$\bar{2}$		$\bar{11}$		$\bar{24}$		$\bar{14}$		$\bar{3}$		3		8		3		1	
a^{13}		17		17		2		$\bar{12}$		$\bar{27}$		$\bar{21}$		$\bar{6}$		3		11		9		3		$\bar{5}$
a^{14}	15		39		21		6		$\bar{13}$		$\bar{26}$		$\bar{13}$		2		13		10		8		$\bar{7}$	
a^{15}		42		24		10		$\bar{28}$		$\bar{45}$		$\bar{52}$		$\bar{17}$		$\bar{5}$		13		11		$\bar{5}$		$\bar{3}$
a^{16}	44		41		31		$\bar{15}$		$\bar{33}$		$\bar{59}$		$\bar{26}$		8		28		31		13		$\bar{2}$	
a^{17}		44		28		$\bar{14}$		$\bar{52}$		$\bar{78}$		$\bar{63}$		$\bar{9}$		15		34		18		1		$\bar{18}$
a^{18}	43		77		33		$\bar{5}$		$\bar{35}$		$\bar{63}$		$\bar{11}$		28		51		34		20		$\bar{20}$	
a^{19}		79		32		$\bar{6}$		$\bar{82}$		$\bar{113}$		$\bar{108}$		$\bar{20}$		3		36		19		$\bar{17}$		$\bar{15}$
a^{20}	82		76		43		$\bar{39}$		$\bar{70}$		$\bar{109}$		$\bar{22}$		48		80		69		29		$\bar{13}$	
a^{21}		76		37		$\bar{43}$		$\bar{121}$		$\bar{159}$		$\bar{117}$				36		70		29		$\bar{10}$		$\bar{44}$
a^{22}	76		122		41		$\bar{35}$		$\bar{75}$		$\bar{112}$		6		83		118		76		38		$\bar{45}$	
a^{23}		120		37		$\bar{41}$		$\bar{163}$		$\bar{201}$		$\bar{165}$		$\bar{5}$		31		75		33		$\bar{43}$		$\bar{43}$
a^{24}	122		112		37		$\bar{86}$		$\bar{121}$		$\bar{161}$		2		120		160		123		40		$\bar{40}$	
a^{25}		109		31		$\bar{92}$		$\bar{205}$		$\bar{242}$		$\bar{154}$		39		83		120		37		$\bar{40}$		$\bar{82}$
a^{26}	107		151		25		$\bar{82}$		$\bar{116}$		$\bar{147}$		52		166		203		117		39		$\bar{82}$	
a^{27}		148		25		$\bar{85}$		$\bar{239}$		$\bar{267}$		$\bar{190}$		44		79		113		33		$\bar{84}$		$\bar{76}$
a^{28}	147		125		13		$\bar{136}$		$\bar{161}$		$\bar{188}$		50		206		237		158		37		$\bar{74}$	
a^{29}		121		14		$\bar{137}$		$\bar{265}$		$\bar{286}$		$\bar{152}$		107		135		157		35		$\bar{77}$		$\bar{122}$
a^{30}	119		153		$\bar{1}$		$\bar{123}$		$\bar{141}$		$\bar{151}$		111		243		263		137		28		$\bar{124}$	
a^{31}		149		1		$\bar{123}$		$\bar{281}$		$\bar{286}$		$\bar{165}$		108		123		138		27		$\bar{127}$		$\bar{107}$
a^{32}	147		112		$\bar{15}$		$\bar{167}$		$\bar{169}$		$\bar{164}$		109		270		280		166		13		$\bar{108}$	

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{33}		108		$\overline{13}$		$\overline{166}$		$\overline{280}$		$\overline{270}$		$\overline{109}$		164		169		167		15		$\overline{112}$		$\overline{147}$
a^{34}	107		127		$\overline{27}$		$\overline{138}$		$\overline{123}$		$\overline{108}$		165		286		281		123		$\overline{1}$		$\overline{149}$	
a^{35}		124		$\overline{28}$		$\overline{137}$		$\overline{263}$		$\overline{243}$		$\overline{111}$		151		141		123		1		$\overline{153}$		$\overline{119}$
a^{36}	122		77		$\overline{35}$		$\overline{157}$		$\overline{135}$		$\overline{107}$		152		286		265		137		$\overline{14}$		$\overline{121}$	
a^{37}		74		$\overline{37}$		$\overline{158}$		$\overline{237}$		$\overline{206}$		$\overline{50}$		188		161		136		$\overline{13}$		$\overline{125}$		$\overline{147}$
a^{38}	76		84		$\overline{33}$		$\overline{113}$		$\overline{79}$		$\overline{44}$		190		267		239		85		$\overline{25}$		$\overline{148}$	
a^{39}		82		$\overline{39}$		$\overline{117}$		$\overline{203}$		$\overline{166}$		$\overline{52}$		147		116		82		$\overline{25}$		$\overline{151}$		$\overline{107}$
a^{40}	82		40		$\overline{37}$		$\overline{120}$		$\overline{83}$		$\overline{39}$		154		242		205		92		$\overline{31}$		$\overline{109}$	
a^{41}		40		$\overline{40}$		$\overline{123}$		$\overline{160}$		$\overline{120}$		$\overline{2}$		161		121		86		$\overline{37}$		$\overline{112}$		$\overline{122}$
a^{42}	43		43		$\overline{33}$		$\overline{75}$		$\overline{31}$		5		165		201		163		41		$\overline{37}$		$\overline{120}$	
a^{43}		45		$\overline{38}$		$\overline{76}$		$\overline{118}$		$\overline{83}$		$\overline{6}$		112		75		35		$\overline{41}$		$\overline{122}$		$\overline{76}$
a^{44}	44		10		$\overline{29}$		$\overline{70}$		$\overline{36}$				117		159		121		43		$\overline{37}$		$\overline{76}$	
a^{45}		13		$\overline{29}$		$\overline{69}$		$\overline{80}$		$\overline{48}$		22		109		70		39		$\overline{43}$		$\overline{76}$		$\overline{82}$
a^{46}	15		17		$\overline{19}$		$\overline{36}$		$\overline{3}$		20		108		113		82		6		$\overline{32}$		$\overline{79}$	
a^{47}		20		$\overline{20}$		$\overline{34}$		$\overline{51}$		$\overline{28}$		11		63		35		5		$\overline{33}$		$\overline{77}$		$\overline{43}$
a^{48}	18		$\overline{1}$		$\overline{18}$		$\overline{34}$		$\overline{15}$		9		63		78		52		14		$\overline{28}$		$\overline{44}$	
a^{49}		2		$\overline{13}$		$\overline{31}$		$\overline{28}$		$\overline{8}$		26		59		33		15		$\overline{31}$		$\overline{41}$		$\overline{44}$
a^{50}	3		5		$\overline{11}$		$\overline{13}$		5		17		52		45		28		$\overline{10}$		$\overline{24}$		$\overline{42}$	
a^{51}		7		$\overline{8}$		$\overline{10}$		$\overline{13}$		$\overline{2}$		13		26		13		$\overline{6}$		$\overline{21}$		$\overline{39}$		$\overline{15}$
a^{52}	5		$\overline{3}$		$\overline{9}$		$\overline{11}$		$\overline{3}$		6		21		27		12		$\overline{2}$		$\overline{17}$		$\overline{17}$	
a^{53}		$\overline{1}$		$\overline{3}$		$\overline{8}$		$\overline{3}$		3		14		24		11		2		$\overline{15}$		$\overline{14}$		$\overline{18}$
a^{54}	$\overline{1}$		1		$\overline{4}$		$\overline{2}$		3		5		16		10		2		$\overline{9}$		$\overline{11}$		$\overline{17}$	
a^{55}		2		$\overline{1}$		1		$\overline{1}$		2		7		7		3		$\overline{5}$		$\overline{5}$		$\overline{15}$		$\overline{3}$
a^{56}	1		$\overline{1}$		$\overline{2}$		$\overline{3}$		$\overline{3}$		1		2		4		$\overline{1}$		$\overline{2}$		$\overline{8}$		$\overline{5}$	
a^{57}		$\overline{1}$				$\overline{2}$		$\overline{1}$		3		5		7		3		2		$\overline{4}$		$\overline{3}$		$\overline{5}$
a^{58}			1						2		2		3		1		$\overline{1}$		$\overline{5}$		$\overline{5}$		$\overline{5}$	
a^{59}				$\overline{1}$		$\overline{1}$		$\overline{1}$										$\overline{3}$		$\overline{1}$		$\overline{4}$		1
a^{60}	1						1		1		2		1		2				$\overline{1}$		$\overline{2}$			
a^{61}		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				$\overline{2}$		$\overline{1}$		$\overline{2}$				$\overline{1}$
a^{62}					1		1		2		2		2		1		1							
a^{63}										$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		
a^{64}																	1		1		1		1	
a^{65}																								$\overline{1}$

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{10})(1-a^{12})^2(1-a^{14})(1-a^{16})(1-a^2x^6)$
$(1-a^2x^{10})(1-a^2x^{14})(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^0	1																			
a^3				1		1		1		2		1		1		1		1		
a^4	1				2		2		3		2		2		2		1		1	
a^5		1		3		4		4		3		4		2		2				
a^6			4		4		7		7		5		6		1		2			
a^7		4		8		9		10		11		7		6		2				
a^8	5		8		13		16		16		14		7		6		1		$\overline{1}$	
a^9		10		17		20		22		19		15		7		1		$\overline{3}$		$\overline{7}$
a^{10}	4		20		25		30		33		20		13		2		$\overline{3}$		$\overline{10}$	
a^{11}		21		32		41		43		40		20		11		$\overline{4}$		$\overline{14}$		$\overline{13}$
a^{12}	17		35		50		60		57		37		16				$\overline{18}$		$\overline{25}$	
a^{13}		39		57		75		71		57		28		6		$\overline{29}$		$\overline{34}$		$\overline{41}$
a^{14}	20		64		86		90		92		44		13		$\overline{31}$		$\overline{46}$		$\overline{59}$	
a^{15}		67		94		121		108		96		23		$\overline{11}$		$\overline{63}$		$\overline{73}$		$\overline{79}$
a^{16}	47		103		135		143		135		57		$\overline{7}$		$\overline{65}$		$\overline{91}$		$\overline{117}$	
a^{17}		108		142		181		154		116		3		$\overline{45}$		$\overline{139}$		$\overline{136}$		$\overline{148}$
a^{18}	61		152		195		191		181		37		$\overline{43}$		$\overline{149}$		$\overline{176}$		$\overline{198}$	
a^{19}		157		201		257		199		149		$\overline{38}$		$\overline{104}$		$\overline{239}$		$\overline{221}$		$\overline{222}$
a^{20}	97		211		270		260		225		21		$\overline{107}$		$\overline{252}$		$\overline{271}$		$\overline{302}$	
a^{21}		215		273		339		239		157		$\overline{108}$		$\overline{200}$		$\overline{391}$		$\overline{330}$		$\overline{338}$
a^{22}	120		281		348		308		262		$\overline{42}$		$\overline{206}$		$\overline{412}$		$\overline{410}$		$\overline{434}$	
a^{23}		284		348		418		269		159		$\overline{215}$		$\overline{327}$		$\overline{562}$		$\overline{462}$		$\overline{440}$

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
																				a^0
	1																			a^3
		1																		a^4
			$\overline{2}$				$\overline{1}$				$\overline{1}$									a^5
		$\overline{1}$		$\overline{1}$				$\overline{1}$												a^6
	$\overline{3}$				$\overline{1}$								1				1			a^7
$\overline{3}$		$\overline{4}$		$\overline{1}$		$\overline{2}$				1										a^8
	$\overline{4}$		$\overline{6}$		$\overline{3}$		$\overline{1}$				1				1				$\overline{1}$	a^9
$\overline{11}$		$\overline{9}$		$\overline{7}$		$\overline{2}$						1		1						a^{10}
	$\overline{16}$		$\overline{11}$		$\overline{6}$				2		2		3							a^{11}
$\overline{23}$		$\overline{24}$		$\overline{9}$		$\overline{4}$		1		3		5		2						a^{12}
	$\overline{36}$		$\overline{29}$		$\overline{9}$				4		7		2		2				$\overline{1}$	a^{13}
$\overline{55}$		$\overline{46}$		$\overline{20}$		$\overline{4}$		7		9		11		4		1		$\overline{1}$		a^{14}
	$\overline{65}$		$\overline{40}$		$\overline{9}$		8		20		15		12		4					a^{15}
$\overline{89}$		$\overline{78}$		$\overline{20}$				27		24		23		9		1		$\overline{4}$		a^{16}
	$\overline{102}$		$\overline{74}$		$\overline{5}$		25		38		30		17		7		$\overline{4}$		$\overline{5}$	a^{17}
$\overline{147}$		$\overline{121}$		$\overline{23}$		19		57		41		45		13				$\overline{10}$		a^{18}
	$\overline{150}$		$\overline{87}$		25		57		83		55		39		6		$\overline{8}$		$\overline{4}$	a^{19}
$\overline{202}$		$\overline{164}$		9		50		112		83		74		16		$\overline{3}$		$\overline{21}$		a^{20}
	$\overline{194}$		$\overline{113}$		63		109		137		86		48		6		$\overline{19}$		$\overline{17}$	a^{21}
$\overline{276}$		$\overline{202}$		43		107		194		121		112		16		$\overline{11}$		$\overline{39}$		a^{22}
	$\overline{230}$		$\overline{102}$		149		194		232		126		81		$\overline{2}$		$\overline{34}$		$\overline{20}$	a^{23}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{24}	165		353		419		366		278		$\overline{122}$		$\overline{338}$		$\overline{586}$		$\overline{555}$		$\overline{569}$	
a^{25}		353		417		490		275		115		$\overline{356}$		$\overline{481}$		$\overline{777}$		$\overline{593}$		$\overline{551}$
a^{26}	189		415		484		386		269		$\overline{247}$		$\overline{496}$		$\overline{800}$		$\overline{716}$		$\overline{692}$	
a^{27}		413		478		544		254		68		$\overline{519}$		$\overline{652}$		$\overline{976}$		$\overline{708}$		$\overline{622}$
a^{28}	223		471		529		403		235		$\overline{374}$		$\overline{669}$		$\overline{996}$		$\overline{839}$		$\overline{794}$	
a^{29}		464		521		570		211		$\overline{22}$		$\overline{694}$		$\overline{821}$		$\overline{1181}$		$\overline{795}$		$\overline{671}$
a^{30}	241		506		551		375		171		$\overline{530}$		$\overline{840}$		$\overline{1186}$		$\overline{959}$		$\overline{844}$	
a^{31}		499		538		568		139		$\overline{120}$		$\overline{859}$		$\overline{978}$		$\overline{1326}$		$\overline{832}$		$\overline{649}$
a^{32}	254		521		541		332		87		$\overline{669}$		$\overline{988}$		$\overline{1327}$		$\overline{998}$		$\overline{839}$	
a^{33}		510		529		534		49		$\overline{224}$		$\overline{1007}$		$\overline{1088}$		$\overline{1420}$		$\overline{809}$		$\overline{584}$
a^{34}	254		508		508		260		$\overline{5}$		$\overline{792}$		$\overline{1098}$		$\overline{1401}$		$\overline{991}$		$\overline{773}$	
a^{35}		499		492		474		$\overline{42}$		$\overline{322}$		$\overline{1104}$		$\overline{1144}$		$\overline{1432}$		$\overline{729}$		$\overline{459}$
a^{36}	241		475		449		183		$\overline{101}$		$\overline{877}$		$\overline{1143}$		$\overline{1406}$		$\overline{915}$		$\overline{650}$	
a^{37}		464		435		399		$\overline{132}$		$\overline{398}$		$\overline{1144}$		$\overline{1137}$		$\overline{1376}$		$\overline{593}$		$\overline{297}$
a^{38}	223		419		380		97		$\overline{184}$		$\overline{905}$		$\overline{1133}$		$\overline{1335}$		$\overline{788}$		$\overline{483}$	
a^{39}		413		367		311		$\overline{205}$		$\overline{446}$		$\overline{1122}$		$\overline{1076}$		$\overline{1240}$		$\overline{423}$		$\overline{128}$
a^{40}	189		357		297		16		$\overline{240}$		$\overline{891}$		$\overline{1062}$		$\overline{1203}$		$\overline{619}$		$\overline{306}$	
a^{41}		353		288		222		$\overline{251}$		$\overline{456}$		$\overline{1049}$		$\overline{956}$		$\overline{1051}$		$\overline{250}$		47
a^{42}	165		284		217		$\overline{40}$		$\overline{272}$		$\overline{825}$		$\overline{940}$		$\overline{1011}$		$\overline{441}$		$\overline{121}$	
a^{43}		284		210		147		$\overline{274}$		$\overline{446}$		$\overline{923}$		$\overline{801}$		$\overline{844}$		$\overline{80}$		191
a^{44}	120		213		147		$\overline{88}$		$\overline{278}$		$\overline{728}$		$\overline{780}$		$\overline{818}$		$\overline{264}$		34	
a^{45}		215		146		85		$\overline{270}$		$\overline{386}$		$\overline{769}$		$\overline{630}$		$\overline{619}$		65		297
a^{46}	97		152		91		$\overline{101}$		$\overline{256}$		$\overline{588}$		$\overline{615}$		$\overline{599}$		$\overline{107}$		145	
a^{47}		157		94		36		$\overline{242}$		$\overline{333}$		$\overline{604}$		$\overline{465}$		$\overline{427}$		158		338
a^{48}	61		102		46		$\overline{112}$		$\overline{219}$		$\overline{468}$		$\overline{452}$		$\overline{422}$		$\overline{7}$		215	
a^{49}		108		52		5		$\overline{203}$		$\overline{255}$		$\overline{446}$		$\overline{317}$		$\overline{253}$		209		359
a^{50}	47		62		17		$\overline{91}$		$\overline{175}$		$\overline{333}$		$\overline{309}$		$\overline{258}$		76		243	
a^{51}		67		25		$\overline{10}$		$\overline{158}$		$\overline{192}$		$\overline{307}$		$\overline{196}$		$\overline{136}$		224		321

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
$\overline{321}$		$\overline{224}$		136		196		307		192		158		10		$\overline{25}$		$\overline{67}$		a^{24}
	$\overline{243}$		$\overline{76}$		258		309		333		175		91		$\overline{17}$		$\overline{62}$		$\overline{47}$	a^{25}
$\overline{359}$		$\overline{209}$		253		317		446		255		203		$\overline{5}$		$\overline{52}$		$\overline{108}$		a^{26}
	$\overline{215}$		7		422		452		468		219		112		$\overline{46}$		$\overline{102}$		$\overline{61}$	a^{27}
$\overline{338}$		$\overline{158}$		427		465		604		333		242		$\overline{36}$		$\overline{94}$		$\overline{157}$		a^{28}
	$\overline{145}$		107		599		615		588		256		101		$\overline{91}$		$\overline{152}$		$\overline{97}$	a^{29}
$\overline{297}$		$\overline{65}$		619		630		769		386		270		$\overline{85}$		$\overline{146}$		$\overline{215}$		a^{30}
	$\overline{34}$		264		818		780		728		278		88		$\overline{147}$		$\overline{213}$		$\overline{120}$	a^{31}
$\overline{191}$		80		844		801		923		446		274		$\overline{147}$		$\overline{210}$		$\overline{284}$		a^{32}
	121		441		1011		940		825		272		40		$\overline{217}$		$\overline{284}$		$\overline{165}$	a^{33}
$\overline{47}$		250		1051		956		1049		456		251		$\overline{222}$		$\overline{288}$		$\overline{353}$		a^{34}
	306		619		1203		1062		891		240		$\overline{16}$		$\overline{297}$		$\overline{357}$		$\overline{189}$	a^{35}
128		423		1240		1076		1122		446		205		$\overline{311}$		$\overline{367}$		$\overline{413}$		a^{36}
	483		788		1335		1133		905		184		$\overline{97}$		$\overline{380}$		$\overline{419}$		$\overline{223}$	a^{37}
297		593		1376		1137		1144		398		132		$\overline{399}$		$\overline{435}$		$\overline{464}$		a^{38}
	650		915		1406		1143		877		101		$\overline{183}$		$\overline{449}$		$\overline{475}$		$\overline{241}$	a^{39}
459		729		1432		1144		1104		322		42		$\overline{474}$		$\overline{492}$		$\overline{499}$		a^{40}
	773		991		1401		1098		792		5		$\overline{260}$		$\overline{508}$		$\overline{508}$		$\overline{254}$	a^{41}
584		809		1420		1088		1007		224		$\overline{49}$		$\overline{534}$		$\overline{529}$		$\overline{510}$		a^{42}
	839		998		1327		988		669		$\overline{87}$		$\overline{332}$		$\overline{541}$		$\overline{521}$		$\overline{254}$	a^{43}
649		832		1326		978		859		120		$\overline{139}$		$\overline{568}$		$\overline{538}$		$\overline{499}$		a^{44}
	844		959		1186		840		530		$\overline{171}$		$\overline{375}$		$\overline{551}$		$\overline{506}$		$\overline{241}$	a^{45}
671		795		1181		821		694		22		$\overline{211}$		$\overline{570}$		$\overline{521}$		$\overline{464}$		a^{46}
	794		839		996		669		374		$\overline{235}$		$\overline{403}$		$\overline{529}$		$\overline{471}$		$\overline{223}$	a^{47}
622		708		976		652		519		$\overline{68}$		$\overline{254}$		$\overline{544}$		$\overline{478}$		$\overline{413}$		a^{48}
	692		716		800		496		247		$\overline{269}$		$\overline{386}$		$\overline{484}$		$\overline{415}$		189	a^{49}
551		593		777		481		356		$\overline{115}$		$\overline{275}$		$\overline{490}$		$\overline{417}$		$\overline{353}$		a^{50}
	569		555		586		338		122		$\overline{278}$		$\overline{366}$		$\overline{419}$		$\overline{353}$		$\overline{165}$	a^{51}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{52}	20		34		2		$\bar{81}$		$\bar{126}$		$\bar{232}$		$\bar{194}$		$\bar{149}$		102		230	
a^{53}		39		11		$\bar{16}$		$\bar{112}$		$\bar{121}$		$\bar{194}$		$\bar{107}$		$\bar{43}$		202		276
a^{54}	17		19		$\bar{6}$		$\bar{48}$		$\bar{86}$		$\bar{137}$		$\bar{109}$		$\bar{68}$		113		194	
a^{55}		21		3		$\bar{16}$		$\bar{74}$		$\bar{83}$		$\bar{112}$		$\bar{50}$		$\bar{9}$		164		202
a^{56}	4		8		$\bar{6}$		$\bar{39}$		$\bar{55}$		$\bar{83}$		$\bar{57}$		$\bar{25}$		87		150	
a^{57}		10				$\bar{13}$		$\bar{45}$		$\bar{41}$		$\bar{57}$		$\bar{19}$		23		121		147
a^{58}	5		4		$\bar{7}$		$\bar{17}$		$\bar{30}$		$\bar{38}$		$\bar{25}$		5		74		102	
a^{59}		4		$\bar{1}$		$\bar{9}$		$\bar{23}$		$\bar{24}$		$\bar{27}$				20		78		89
a^{60}					$\bar{4}$		$\bar{12}$		$\bar{15}$		$\bar{20}$		$\bar{8}$		9		40		65	
a^{61}		1		$\bar{1}$		$\bar{4}$		$\bar{11}$		$\bar{9}$		$\bar{7}$		4		20		46		55
a^{62}	1				$\bar{2}$		$\bar{2}$		$\bar{7}$		$\bar{4}$				9		29		36	
a^{63}						$\bar{2}$		$\bar{5}$		$\bar{3}$		$\bar{1}$		4		9		24		23
a^{64}							$\bar{3}$		$\bar{2}$		$\bar{2}$				6		11		16	
a^{65}						$\bar{1}$		$\bar{1}$						2		7		9		11
a^{66}	1				$\bar{1}$				$\bar{1}$				1		3		6		4	
a^{67}										$\bar{1}$				2		1		4		3
a^{68}			$\bar{1}$				$\bar{1}$								1				3	
a^{69}												1				1		1		
a^{70}									1				1				2			
a^{71}																		$\bar{1}$		
a^{72}																			$\bar{1}$	
a^{75}																				

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
440		462		562		327		215		$\overline{159}$		$\overline{269}$		$\overline{418}$		$\overline{348}$		$\overline{284}$		a^{52}
	434		410		412		206		42		$\overline{262}$		$\overline{308}$		$\overline{348}$		$\overline{281}$		$\overline{120}$	a^{53}
338		330		391		200		108		$\overline{157}$		$\overline{239}$		$\overline{339}$		$\overline{273}$		$\overline{215}$		a^{54}
	302		271		252		107		$\overline{21}$		$\overline{225}$		$\overline{260}$		$\overline{270}$		$\overline{211}$		$\overline{97}$	a^{55}
222		221		239		104		38		$\overline{149}$		$\overline{199}$		$\overline{257}$		$\overline{201}$		$\overline{157}$		a^{56}
	198		176		149		43		$\overline{37}$		$\overline{181}$		$\overline{191}$		$\overline{195}$		$\overline{152}$		$\overline{61}$	a^{57}
148		136		139		45		$\overline{3}$		$\overline{116}$		$\overline{154}$		$\overline{181}$		$\overline{142}$		$\overline{108}$		a^{58}
	117		91		65		7		$\overline{57}$		$\overline{135}$		$\overline{143}$		$\overline{135}$		$\overline{103}$		$\overline{47}$	a^{59}
79		73		63		11		$\overline{23}$		$\overline{96}$		$\overline{108}$		$\overline{121}$		$\overline{94}$		$\overline{67}$		a^{60}
	59		46		31		$\overline{13}$		$\overline{44}$		$\overline{92}$		$\overline{90}$		$\overline{86}$		$\overline{64}$		$\overline{20}$	a^{61}
41		34		29		$\overline{6}$		$\overline{28}$		$\overline{57}$		$\overline{71}$		$\overline{75}$		$\overline{57}$		$\overline{39}$		a^{62}
	25		18				$\overline{16}$		$\overline{37}$		$\overline{57}$		$\overline{60}$		$\overline{50}$		$\overline{35}$		$\overline{17}$	a^{63}
13		14		4		$\overline{11}$		$\overline{20}$		$\overline{40}$		$\overline{43}$		$\overline{41}$		$\overline{32}$		$\overline{21}$		a^{64}
	10		3		$\overline{2}$		$\overline{13}$		$\overline{20}$		$\overline{33}$		$\overline{30}$		$\overline{25}$		$\overline{20}$		$\overline{4}$	a^{65}
7		3		$\overline{1}$		$\overline{7}$		$\overline{15}$		$\overline{19}$		$\overline{22}$		$\overline{20}$		$\overline{17}$		$\overline{10}$		a^{66}
	1		$\overline{1}$		$\overline{6}$		$\overline{7}$		$\overline{14}$		$\overline{16}$		$\overline{16}$		$\overline{13}$		$\overline{8}$		$\overline{5}$	a^{67}
				$\overline{2}$		$\overline{6}$		$\overline{7}$		$\overline{11}$		$\overline{10}$		$\overline{9}$		$\overline{8}$		$\overline{4}$		a^{68}
			$\overline{2}$		$\overline{1}$		$\overline{6}$		$\overline{5}$		$\overline{7}$		$\overline{7}$		$\overline{4}$		$\overline{4}$			a^{69}
				$\overline{2}$		$\overline{2}$		$\overline{4}$		$\overline{3}$		$\overline{4}$		$\overline{4}$		$\overline{3}$		$\overline{1}$		a^{70}
	$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{3}$		$\overline{2}$		$\overline{2}$				$\overline{1}$	a^{71}
		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				a^{72}
																			$\overline{1}$	a^{75}

Table of Groundforms.

Degree in the Coefficients.										Order in the Variables.											
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6	.		4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

DECIMIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)$.

Numerator: $1+3a^2+11a^3+27a^4+58a^5+112a^6+193a^7+318a^8+485a^9+699a^{10}+951a^{11}+1245a^{12}+1541a^{13}+1842a^{14}+2108a^{15}+2321a^{16}+2451a^{17}+2506a^{18}+2451a^{19}+2321a^{20}+2108a^{21}+1842a^{22}+1541a^{23}+1245a^{24}+951a^{25}+699a^{26}+485a^{27}+318a^{28}+193a^{29}+112a^{30}+58a^{31}+27a^{32}+11a^{33}+3a^{34}+a^{36}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)$
$(1-a^9)(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{29}	x^{22}	x^{24}	x^{26}	x^{28}
a^0	1														
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$										
a^2	$\bar{1}$		1	1	2	2	2	1	1						
a^3	$\bar{1}$	2	1	2	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$			
a^4		1	2			$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	1	1	1	1	
a^5		2	2			$\bar{1}$	$\bar{2}$		1	1	1	1			$\bar{1}$
a^6	3	1	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$			1		1	$\bar{1}$	$\bar{1}$	
a^7		1		1	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	1
a^8	2	3	4	2	1	2	$\bar{2}$	$\bar{1}$		2				$\bar{1}$	1
a^9	2	5		1	$\bar{2}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{3}$	$\bar{2}$					
a^{10}	4	3	3		$\bar{4}$	$\bar{6}$	$\bar{6}$	$\bar{3}$		4	5	4	2	1	
a^{11}		4	2	$\bar{3}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{4}$	2	2	5	1	1	$\bar{1}$	$\bar{3}$
a^{12}	6	5	4	1	$\bar{2}$	$\bar{5}$	$\bar{7}$	$\bar{2}$	2	5	4	3	1	$\bar{2}$	
a^{13}	$\bar{1}$	3	$\bar{1}$	$\bar{5}$	$\bar{11}$	$\bar{17}$	$\bar{12}$	$\bar{9}$		2	6	3	1	$\bar{1}$	$\bar{2}$
a^{14}	1	4	7	$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	5	10	14	11	7	4	$\bar{3}$	$\bar{2}$
a^{15}		5	1	$\bar{4}$	$\bar{9}$	$\bar{17}$	$\bar{12}$	$\bar{6}$	3	3	5	1	$\bar{4}$	$\bar{5}$	$\bar{4}$
a^{16}	3	1	2	$\bar{5}$	$\bar{11}$	$\bar{11}$	$\bar{6}$	3	10	17	13	8		$\bar{2}$	
a^{17}	$\bar{4}$	1	$\bar{3}$	$\bar{9}$	$\bar{10}$	$\bar{10}$	$\bar{2}$	4	15	13	8	1	$\bar{4}$	$\bar{6}$	$\bar{6}$
a^{18}	4		$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3	13	13	14	4	1	$\bar{6}$	$\bar{8}$	1
a^{19}	$\bar{3}$	$\bar{5}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{7}$	1	2	4	4	1	$\bar{3}$	$\bar{9}$	$\bar{3}$	$\bar{1}$
a^{20}		$\bar{3}$	$\bar{1}$		4	14	13	16	13	14	4		$\bar{1}$	$\bar{3}$	

* **Numerator and denominator divisible by $1-a$; see foot-note to reduced form for sextic.**

Numerator—*Continued.*

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^{21}	$\bar{1}$	$\bar{3}$	$\bar{9}$	$\bar{3}$	1	4	4	2	1	$\bar{7}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{5}$	$\bar{3}$
a^{22}	1	$\bar{8}$	$\bar{6}$	1	4	14	13	13	3	2	$\bar{1}$	$\bar{1}$	$\bar{1}$		4
a^{23}	$\bar{6}$	$\bar{6}$	$\bar{4}$	1	8	13	15	4	$\bar{2}$	$\bar{10}$	$\bar{10}$	$\bar{9}$	$\bar{3}$	1	$\bar{4}$
a^{24}		$\bar{2}$		8	13	17	10	3	$\bar{6}$	$\bar{11}$	$\bar{11}$	$\bar{5}$	2	1	3
a^{25}	$\bar{4}$	$\bar{5}$	$\bar{4}$	1	5	3	3	$\bar{6}$	$\bar{12}$	$\bar{17}$	$\bar{9}$	$\bar{4}$	1	5	
a^{26}	$\bar{2}$	$\bar{3}$	4	7	11	14	10	5	$\bar{5}$	$\bar{6}$	$\bar{3}$	$\bar{1}$	7	4	1
a^{27}	$\bar{2}$	$\bar{1}$	1	3	6	2		$\bar{9}$	$\bar{12}$	$\bar{17}$	$\bar{11}$	$\bar{5}$	$\bar{1}$	3	$\bar{1}$
a^{28}		$\bar{2}$	1	3	4	5	2	$\bar{2}$	$\bar{7}$	$\bar{5}$	$\bar{2}$	1	4	5	6
a^{29}	$\bar{3}$	$\bar{1}$	1	1	5	2	2	$\bar{4}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{3}$	2	4	
a^{30}		1	2	4	5	4		$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{4}$		3	3	4
a^{31}						$\bar{2}$	$\bar{3}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{2}$	1		5	2
a^{32}	1	$\bar{1}$				2		$\bar{1}$	$\bar{2}$	2	1	2	4	3	2
a^{33}	1	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	1		1	
a^{34}		$\bar{1}$	$\bar{1}$	1		1			$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1	3
a^{35}	$\bar{1}$			1	1	1	1		$\bar{2}$	$\bar{1}$			2	2	
a^{36}		1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$			2	1	
a^{37}				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	2	1	2	$\bar{1}$
a^{38}							1	1	2	2	2	1	1		$\bar{1}$
a^{39}											$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{40}															1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)^2(1-a^8)(1-a^9)(1-a^{10})(1-a^{14})$
$(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
a^0	1																								
a^3		1		2	1	1	2	1	1	1	1		1												
a^4			3	1	3	3	2	3	1	2	1	1		1											
a^5		3	3	4	5	4	5	2	4		1		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$						
a^6	2	2	6	8	8	9	6	7	2	4				$\overline{2}$		$\overline{1}$		$\overline{1}$							
a^7		7	10	11	13	11	11	7	6	1			$\overline{4}$		$\overline{2}$				1		1		1		
a^8	4	8	14	18	20	22	12	11	4	2	$\overline{2}$	$\overline{4}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{3}$				1					
a^9	4	15	21	27	30	24	23	12	7	$\overline{1}$	$\overline{8}$	$\overline{6}$	$\overline{9}$	$\overline{5}$	$\overline{5}$	$\overline{1}$	$\overline{1}$		2		1				$\overline{1}$
a^{10}	7	20	31	37	39	39	22	15	2	$\overline{8}$	$\overline{11}$	$\overline{18}$	$\overline{14}$	$\overline{15}$	$\overline{5}$	$\overline{4}$		2	1	2		1			
a^{11}	8	28	41	50	56	46	31	12	$\overline{2}$	$\overline{17}$	$\overline{28}$	$\overline{25}$	$\overline{26}$	$\overline{18}$	$\overline{13}$	$\overline{5}$	1	2	5	1	2		$\overline{1}$		
a^{12}	15	38	54	67	69	60	33	11	$\overline{12}$	$\overline{33}$	$\overline{41}$	$\overline{45}$	$\overline{36}$	$\overline{31}$	$\overline{12}$	$\overline{3}$	2	7	6	7	2	1		$\overline{1}$	
a^{13}	15	49	72	84	90	70	37	3	$\overline{26}$	$\overline{54}$	$\overline{66}$	$\overline{62}$	$\overline{56}$	$\overline{39}$	$\overline{21}$	$\overline{2}$	8	12	14	7	4	1	$\overline{1}$		
a^{14}	20	61	87	104	106	82	32	$\overline{9}$	$\overline{48}$	$\overline{86}$	$\overline{95}$	$\overline{93}$	$\overline{73}$	$\overline{55}$	$\overline{20}$	2	14	20	18	16	5	2	$\overline{3}$	$\overline{3}$	
a^{15}	27	75	108	127	128	92	32	$\overline{26}$	$\overline{76}$	$\overline{120}$	$\overline{134}$	$\overline{119}$	$\overline{100}$	$\overline{66}$	$\overline{25}$	11	27	32	32	18	9		$\overline{2}$	$\overline{2}$	$\overline{2}$
a^{16}	29	90	129	147	146	100	22	$\overline{49}$	$\overline{110}$	$\overline{165}$	$\overline{172}$	$\overline{157}$	$\overline{120}$	$\overline{77}$	$\overline{18}$	26	41	52	44	32	9	1	$\overline{6}$	$\overline{7}$	
a^{17}	35	105	148	168	164	103	5	$\overline{81}$	$\overline{153}$	$\overline{218}$	$\overline{227}$	$\overline{195}$	$\overline{150}$	$\overline{88}$	$\overline{15}$	44	67	70	63	37	14		$\overline{9}$	$\overline{8}$	$\overline{4}$
a^{18}	40	119	168	191	179	105	$\overline{11}$	$\overline{115}$	$\overline{201}$	$\overline{272}$	$\overline{274}$	$\overline{232}$	$\overline{169}$	$\overline{94}$	1	71	93	101	82	51	13	$\overline{6}$	$\overline{13}$	$\overline{15}$	$\overline{4}$
a^{19}	44	132	189	204	192	101	$\overline{36}$	$\overline{154}$	$\overline{254}$	$\overline{330}$	$\overline{330}$	$\overline{267}$	$\overline{190}$	$\overline{88}$	24	108	132	133	112	62	17	$\overline{7}$	$\overline{20}$	$\overline{20}$	$\overline{7}$
a^{20}	47	147	202	221	200	94	$\overline{64}$	$\overline{202}$	$\overline{305}$	$\overline{395}$	$\overline{379}$	$\overline{303}$	$\overline{203}$	$\overline{85}$	48	150	172	171	133	74	14	$\overline{16}$	$\overline{30}$	$\overline{28}$	$\overline{8}$
a^{21}	55	154	216	232	203	83	$\overline{98}$	$\overline{241}$	$\overline{365}$	$\overline{447}$	$\overline{431}$	$\overline{327}$	$\overline{208}$	$\overline{70}$	92	196	222	208	166	85	11	$\overline{26}$	$\overline{41}$	$\overline{38}$	$\overline{15}$
a^{22}	52	164	226	236	202	63	$\overline{127}$	$\overline{292}$	$\overline{413}$	$\overline{506}$	$\overline{470}$	$\overline{346}$	$\overline{210}$	$\overline{42}$	130	257	272	255	194	93	7	$\overline{37}$	$\overline{53}$	$\overline{49}$	$\overline{15}$
a^{23}	57	166	229	237	194	50	$\overline{168}$	$\overline{333}$	$\overline{465}$	$\overline{550}$	$\overline{502}$	$\overline{359}$	$\overline{193}$	$\overline{17}$	186	310	327	296	220	103	$\overline{4}$	$\overline{52}$	$\overline{73}$	$\overline{61}$	$\overline{20}$
a^{24}	56	172	228	236	187	22	$\overline{191}$	$\overline{372}$	$\overline{499}$	$\overline{585}$	$\overline{527}$	$\overline{353}$	$\overline{176}$	28	238	375	380	336	247	104	$\overline{13}$	$\overline{71}$	$\overline{88}$	$\overline{75}$	$\overline{27}$
a^{25}	57	166	227	225	168	7	$\overline{229}$	$\overline{401}$	$\overline{536}$	$\overline{610}$	$\overline{529}$	$\overline{347}$	$\overline{143}$	66	298	433	430	376	266	105	$\overline{31}$	$\overline{89}$	$\overline{109}$	$\overline{90}$	$\overline{29}$
a^{26}	52	164	217	211	155	$\overline{24}$	$\overline{249}$	$\overline{431}$	$\overline{551}$	$\overline{624}$	$\overline{536}$	$\overline{323}$	$\overline{114}$	119	346	487	474	403	281	98	$\overline{46}$	$\overline{111}$	$\overline{131}$	$\overline{105}$	$\overline{35}$
a^{27}	55	154	203	198	130	$\overline{38}$	$\overline{273}$	$\overline{442}$	$\overline{562}$	$\overline{620}$	$\overline{512}$	$\overline{296}$	$\overline{65}$	160	407	537	512	430	286	93	$\overline{68}$	$\overline{134}$	$\overline{150}$	$\overline{119}$	$\overline{40}$
a^{28}	47	147	190	176	112	$\overline{64}$	$\overline{281}$	$\overline{448}$	$\overline{556}$	$\overline{603}$	$\overline{490}$	$\overline{252}$	$\overline{26}$	216	448	578	541	443	296	76	$\overline{87}$	$\overline{155}$	$\overline{169}$	$\overline{132}$	$\overline{44}$

Numerator—*Continued.*

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
a^{29}	44	132	169	155	87	$\overline{76}$	$\overline{296}$	$\overline{443}$	$\overline{541}$	$\overline{578}$	$\overline{448}$	$\overline{216}$	26	252	490	603	556	448	281	64	$\overline{112}$	$\overline{176}$	$\overline{190}$	$\overline{147}$	$\overline{47}$
a^{30}	40	119	150	134	68	$\overline{93}$	$\overline{286}$	$\overline{430}$	$\overline{512}$	$\overline{537}$	$\overline{407}$	$\overline{160}$	65	296	512	620	562	442	273	38	$\overline{130}$	$\overline{198}$	$\overline{203}$	$\overline{154}$	$\overline{55}$
a^{31}	35	105	131	111	46	$\overline{98}$	$\overline{281}$	$\overline{403}$	$\overline{474}$	$\overline{487}$	$\overline{346}$	$\overline{119}$	114	323	536	624	551	431	249	24	$\overline{155}$	$\overline{211}$	$\overline{217}$	$\overline{164}$	$\overline{52}$
a^{32}	29	90	109	89	31	$\overline{105}$	$\overline{266}$	$\overline{376}$	$\overline{430}$	$\overline{433}$	$\overline{298}$	$\overline{66}$	143	347	529	610	536	401	229	$\overline{7}$	$\overline{168}$	$\overline{225}$	$\overline{227}$	$\overline{166}$	$\overline{57}$
a^{33}	27	75	88	71	13	$\overline{104}$	$\overline{247}$	$\overline{336}$	$\overline{380}$	$\overline{375}$	$\overline{238}$	$\overline{28}$	176	353	527	585	499	372	191	$\overline{22}$	$\overline{187}$	$\overline{236}$	$\overline{228}$	$\overline{172}$	$\overline{56}$
a^{34}	20	61	73	52	4	$\overline{103}$	$\overline{220}$	$\overline{296}$	$\overline{327}$	$\overline{310}$	$\overline{186}$	17	193	359	502	550	465	333	168	$\overline{50}$	$\overline{194}$	$\overline{237}$	$\overline{229}$	$\overline{166}$	$\overline{57}$
a^{35}	15	49	53	37	$\overline{7}$	$\overline{93}$	$\overline{194}$	$\overline{255}$	$\overline{272}$	$\overline{257}$	$\overline{130}$	42	210	346	470	506	413	292	127	$\overline{63}$	$\overline{202}$	$\overline{236}$	$\overline{226}$	$\overline{164}$	$\overline{52}$
a^{36}	15	38	41	26	$\overline{11}$	$\overline{85}$	$\overline{166}$	$\overline{208}$	$\overline{222}$	$\overline{196}$	$\overline{92}$	70	208	327	431	447	365	241	98	$\overline{83}$	$\overline{203}$	$\overline{232}$	$\overline{216}$	$\overline{154}$	$\overline{55}$
a^{37}	8	28	30	16	$\overline{14}$	$\overline{74}$	$\overline{133}$	$\overline{171}$	$\overline{172}$	$\overline{150}$	$\overline{48}$	85	203	303	379	395	305	202	64	$\overline{94}$	$\overline{200}$	$\overline{221}$	$\overline{202}$	$\overline{147}$	$\overline{47}$
a^{38}	7	20	20	7	$\overline{17}$	$\overline{62}$	$\overline{112}$	$\overline{133}$	$\overline{132}$	$\overline{108}$	$\overline{24}$	88	190	267	330	330	254	154	36	$\overline{101}$	$\overline{192}$	$\overline{204}$	$\overline{189}$	$\overline{132}$	$\overline{44}$
a^{39}	4	15	13	6	$\overline{13}$	$\overline{51}$	$\overline{82}$	$\overline{101}$	$\overline{93}$	$\overline{71}$	$\overline{1}$	94	169	232	274	272	201	115	11	$\overline{105}$	$\overline{179}$	$\overline{191}$	$\overline{168}$	$\overline{119}$	$\overline{40}$
a^{40}	4	8	9		$\overline{14}$	$\overline{37}$	$\overline{63}$	$\overline{70}$	$\overline{67}$	$\overline{44}$	15	88	150	195	227	218	153	81	$\overline{5}$	$\overline{103}$	$\overline{164}$	$\overline{168}$	$\overline{148}$	$\overline{105}$	$\overline{35}$
a^{41}		7	6	$\overline{1}$	$\overline{9}$	$\overline{32}$	$\overline{44}$	$\overline{52}$	$\overline{41}$	$\overline{26}$	18	77	120	157	172	165	110	49	$\overline{22}$	$\overline{100}$	$\overline{146}$	$\overline{147}$	$\overline{129}$	$\overline{90}$	$\overline{29}$
a^{42}	2	2	2		$\overline{9}$	$\overline{18}$	$\overline{32}$	$\overline{32}$	$\overline{27}$	$\overline{11}$	25	66	100	119	134	120	76	26	$\overline{32}$	$\overline{92}$	$\overline{128}$	$\overline{127}$	$\overline{108}$	$\overline{75}$	$\overline{27}$
a^{43}		3	3	$\overline{2}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{20}$	$\overline{14}$	$\overline{2}$	20	55	73	93	95	86	48	9	$\overline{32}$	$\overline{82}$	$\overline{106}$	$\overline{104}$	$\overline{87}$	$\overline{61}$	$\overline{20}$
a^{44}			1	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{14}$	$\overline{12}$	$\overline{8}$	2	21	39	56	62	66	54	26	$\overline{3}$	$\overline{37}$	$\overline{70}$	$\overline{90}$	$\overline{84}$	$\overline{72}$	$\overline{49}$	$\overline{15}$
a^{45}		1		$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	3	12	31	36	45	41	33	12	$\overline{11}$	$\overline{33}$	$\overline{60}$	$\overline{69}$	$\overline{67}$	$\overline{54}$	$\overline{38}$	$\overline{15}$
a^{46}			1		$\overline{2}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{1}$	5	13	18	26	25	28	17	2	$\overline{12}$	$\overline{31}$	$\overline{46}$	$\overline{56}$	$\overline{50}$	$\overline{41}$	$\overline{28}$	$\overline{8}$
a^{47}				$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{2}$		4	5	15	14	18	11	8	$\overline{2}$	$\overline{15}$	$\overline{22}$	$\overline{39}$	$\overline{39}$	$\overline{37}$	$\overline{31}$	$\overline{20}$	$\overline{7}$
a^{48}	1				$\overline{1}$		$\overline{2}$		1	1	5	5	9	6	8	1	$\overline{7}$	$\overline{12}$	$\overline{23}$	$\overline{24}$	$\overline{30}$	$\overline{27}$	$\overline{21}$	$\overline{15}$	$\overline{4}$
a^{49}						$\overline{1}$				3	1	6	3	4	2	$\overline{2}$	$\overline{4}$	$\overline{11}$	$\overline{12}$	$\overline{22}$	$\overline{20}$	$\overline{18}$	$\overline{14}$	$\overline{8}$	$\overline{4}$
a^{50}			$\overline{1}$		$\overline{1}$		$\overline{1}$				2		4			$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{11}$	$\overline{11}$	$\overline{13}$	$\overline{11}$	$\overline{10}$	$\overline{7}$	
a^{51}								1		1		2				$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{9}$	$\overline{8}$	$\overline{8}$	$\overline{6}$	$\overline{2}$	$\overline{2}$
a^{52}							1		1		2		1		$\overline{1}$		$\overline{4}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{3}$	
a^{53}												$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{3}$		
a^{54}													$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$	
a^{57}																									$\overline{1}$

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	16	18	20	22	24	26
1						1								
2	1		1		1		1		1					
3		1		2	1	1	2	1	1	1	1		1	
4	1		3	1	3	3	2	3	1	2	1	1		1
5		3	3	4	5	4	5	2	4		1			
6	4	2	5	8	6	8	2	3						
7		7	10	8	12	2	3							
8	5	8	11	15	4	5								
9	5	13	19	8	4									
10	8	20	12	10										
11	8	18	21											
12	12	30												
13	15	16												
14	13	17												
15	19													
16	5													
17	3													

The total number of irreducible invariants and covariants for the first 10 orders (counting in the absolute constant and the quantic itself), it appears from what precedes, is as follows:

Order of Quantic:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Number of Groundforms:	1,	2,	3,	5,	6,	24,	27,	125,	70,	416,	476.

For the benefit of those new to the subject, it may be well to recall the immediate algebraical meaning of either form of the generating function to a binary quantic $(x, y)^n$.

Suppose n an odd number, say 5, then if

$$\frac{1-x^{-2}}{(1-ax^{-5})(1-ax^{-3})(1-ax^{-1})(1-ax)(1-ax^{3})(1-ax^{5})}$$

is expanded in a *bivergent* series, (*i. e.*, one going, as regards the powers of x, in two directions towards infinity,) either generating function of the tables for the quintic is the sum of the terms which contain no negative powers of x. So if n be an even number, say 6,

$$\frac{1-x^{-2}}{(1-ax^{-6})(1-ax^{-4})(1-ax^{-2})(1-a)(1-ax^{2})(1-ax^{4})(1-ax^{6})}$$

being similarly expanded, either generating function of the tables for the sextic is, as before, the sum of the terms which contain only positive or zero powers of x. And so in general, for $(x, y)^n$, the numerator of the so-called *crude* generating function, being always $1-x^{-2}$ and its denominator a product of factors of the form $1-ax^{n-2i}$ (where i takes all values from nought up to n inclusive.) Either generating function of the tables for the n[ic] is the algebraic equivalent of the *positive* branch of the corresponding bivergent series, (that in which only positive powers of x appear,) *plus* the *neutral* branch or term, viz., that which contains neither positive nor negative powers of x, or, which is the same thing, is a function only of a.

I subjoin a few reflexions which appear to me to be desirable on the foregoing tables.

It is scarcely necessary to state, that, in the development of the generating function, whether reduced or representative, the coefficient of $a^m x^\mu$ is the total number of linearly independent covariants of the degree m in the coefficients and the order μ in the variables.

Mr. Franklin will probably, in a future number of the Journal, draw up a statement of the mode in which the tables have been calculated and the precautions taken to insure accuracy;* as regards the reduced form, three methods have been employed in calculating it, viz., Mr. Sylvester's first method, Professor Cayley's method, fully explained in a preceding number of the Journal by its eminent author, and Mr. Sylvester's second method,

* In especial I wish to single out an ingenious device of Mr. Franklin to check the operation of tamisage by introducing a common superfluous factor into the numerator and denominator of the representative generating function so selected as that the augmented denominator shall not cease to be representative; the effect of this will be to cause the groundforms obtained by tamisage of the augmented numerator to be the same as before, except that the groundform represented by the additional factor will not be found among them.

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much briefer than his other, but, in general, not so brief as Professor Cayley's, which last, however, involves a delicate point in the expansion of series, the assumed principle of which, although its validity on moral grounds of evidence is unquestionable, cannot be regarded as *a priori* self-evident.*

The theory of the generating function, alike for single and simultaneous forms, depends on the law for determining the number of linearly independent in- and co-variants of given order and degree or degrees belonging to a given quantic or system of quantics, a proof of which will be found at the end of a memoir by Mr. Sylvester in Borchardt's Journal, and also in the London and Edinburgh Philosophical Magazine, that leaves nothing to be desired as regards rigor of demonstration. The law itself for the case of a single quantic was first stated by Professor Cayley whilst the theory was still in its infancy.

But besides this fundamental theorem, in order to deduce the tables of groundforms, a *fundamental postulate* still awaiting demonstration is necessary, which is, that no more linear relations between in- or co-variants are to be supposed to exist than are necessary in order to satisfy the *fundamental theorem.* The application of this principle in such a mode as to substitute a finite for an infinite process, leads to the use of representative generating functions and the simplified method of *tamisage.* The validity of the fundamental-postulate which is in accord with the law of parcimony is verified by its conducting to results which have been proved to be accurate for single binary quantics up to the sixth order inclusive, for pairs of binary quantics up to the fourth order inclusive, and also for systems of an indefinite number of linear and quadratic binary forms.†

The application of this principle discloses the remarkable singularity that for the quantic of the seventh order, there exists no finite representative generating function as shown in what follows.

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*In Prof. Cayley's method the crude generating function is regarded as a function of a; in my two methods as a function of x.

†If the *fundamental postulate* were called into question, this (it may be proved) would not affect the fact of the existence of the groundforms obtained by its aid, but only the possibility of the existence of other groundforms over and above those so obtained. Thus my tables of groundforms could only err (were that possible, which I do not believe it to be) in defect; and as those found by the German method can only err in excess, it follows that, whenever the tables coincide, both must be correct. The tables of groundforms here given, up to the sixth order, inclusive, and all those that follow, coincide exactly with those obtained by Clebsch, Gordan and Gundelfinger, when these latter are rectified by the omission of certain supposed groundforms which, in the Comptes Rendus, I have conclusively proved to be composite.

The invariantive part of the numerator of the reduced form for the seventhic is

$$1-a^6+2a^8-a^{10}+5a^{12}+2a^{14}+6a^{16}+2a^{18}+5a^{20}-a^{22}+2a^{24}-a^{26}+a^{32},$$

and the invariantive part of the denominator is $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})$. Multiplying numerator and denominator by $(1+a^6)$, their invariantive portions* become, respectively,

$$1+2a^8-a^{10}+4a^{12}+4a^{14}+5a^{16}+7a^{18}+7a^{20}+5a^{22}+4a^{24}+4a^{26}-a^{28}+2a^{30}+a^{38},$$

and $(1-a^4)(1-a^8)(1-a^{10})(1-a^{12})$.

The factors of the denominator are now, with the exception of $1-a^{10}$, representative factors; $1-a^{10}$ is not such, as a^{10} occurs in the numerator with the coefficient -1. If we multiply numerator and denominator by $1+a^{10}$, the factor $1-a^{20}$ will take the place of $1-a^{10}$ in the denominator, and the numerator will become

$$1+2x^8+4a^{12}+4a^{14}+5a^{16}+9a^{18}+6a^{20}+\ldots\ldots$$

Here the coefficient of a^{20} is not negative, but it is less than the number (8) obtained by composition from the terms $2a^8$ and $4a^{12}$; hence, by the fundamental postulate there is no irreducible invariant of the degree 20. If, instead of multiplying numerator and denominator by $1+a^{10}$, we multiply them by the infinite series $1+a^{10}+a^{20}+\ldots$, the denominator becomes representative and the invariantive part of the numerator becomes the *recurrent* series given in the table (p. 228), in which the coefficient of a^{30}, a^{40} and, in general, all powers of a whose exponents are multiples of and greater than 20, is 9; but 9 is less than the number obtained in the composition of a^{30}, a^{40} (and *a fortiori* of a^{50}, a^{60}, ...) out of the preceding terms; therefore, by the fundamental postulate, there is no irreducible invariant whose degree is any multiple of 10. It is a remarkable and significant fact that in this case the erroneous assumption of $1-a^{10}$ being a representative factor in the denominator of the complete generating function will be found to lead to no subsequent further error in the determination of the other groundforms of the seventhic.

A chorographical law obtains in the numerical tables of the numerators of the representative forms, which plays a considerable part in the complete theory of tamisage, and is too important to be passed over without notice, viz: it will be seen that all these tables consist of a small number of irregular but

* The factors in the denominator which involve x never offer any difficulty, as they represent the given quantic along with the complete system of covariants of the second degree, the several orders of which follow a well known rule.

continuous bands or blocks of alternately positive and negative coefficients which can be drawn asunder without tearing or leaving any hole in the paper.* For the first four orders there is but one such block, for the quintic and the sextic two, for the seventhic five, for the octavic three, and for the 9^{ic} and 10^{ic} four. A similar law obtains for systems of quantics, as for instance in the case of two simultaneous quantics, the corresponding tables consist of detachable solid blocks, alternately positive and negative, and small in number in comparison with the number of terms which they contain, as will be seen in the tables to appear in the next number of the Journal which will contain a complete set of them for all the systems that can be formed of two binary quantics of orders, m, n where neither m nor n exceeds 4.

It is my duty to state that the expense of calculating the tables for quantics of the 7th, 8th, 9th and 10th orders, has been defrayed out of a grant made by the British Association for the Advancement of Science, and I have pleasure in returning my thanks to that distinguished body for this act of aid in enabling me to bring to a successful issue an undertaking of such unusual magnitude and of such pith and moment to the progress of Algèbraical Theory.

* In the operation of tamisage on the numerator of the representative groundforms the terms of the negative blocks are disregarded. In every case treated in these tables, and those to follow in the next number of the Journal, the only surviving terms will be found to be comprised in the first block. Had it turned out otherwise it would have been necessary to ascertain whether the surviving terms belonging to the other odd-numbered blocks would survive the operation of tamisage performed on the infinite aggregate of terms obtained by the development of the generating function; if not, they would have to be rejected. This is what I have found actually happens in a system of quadratic or linear forms when a sufficient number of such forms is employed. In that case, terms not confined to the first block emerge from the tamisage of the numerator of the representative groundforms, but disappear when the tamisage is performed on the infinite aggregate of terms of which the groundform is the sum. Such aggregate, it may be noticed, (I have proved elsewhere,) consists exclusively of positive terms, the coefficients corresponding to non-existing types being always zero and never negative. It is very likely to be found true hereafter that in no case need any, except the first block of terms in the numerator of the representative groundforms, be submitted to tamisage in order to obtain the groundforms not represented in the denominator, and so in like manner that, in order to obtain the ground-syzygies of the first kind, *i. e.* those that concern the groundforms, only the first positive and the first negative block need be considered, and so on for syzygies of the higher orders, each time a new block being taken into account until all are exhausted, it being quite conceivable that the number of blocks may designate the highest order of syzygy that occurs in any case, subject in the case of a linear or quadratic form (for which the block reduces to a single term, viz: unity) to the obvious exception that, for them, the syzygies become abortive.

To explain what is meant by syzygies of successive orders, suppose Z to be a rational and integral function of groundforms which, regarded as a function of the coefficients, is identically zero, then $Z=0$ is a syzygy and Z may be termed a syzygant of the first order and, if incapable of being resolved into a sum of products of syzygants multiplied respectively by rational algebraic functions of the groundforms, will be an irreducible or ground-syzygy of the first order. In like manner, if Z' is a function of ground syzygants which, regarded as a function of the groundforms, vanishes identically $Z'=0$ is a syzygy and Z' is a counter-syzygant or a syzygant of the second order, and, if incapable of representation as a sum of products of other syzygants of the second order multiplied respectively by rational integral functions of syzygants of the first order, is a ground-syzygant of the second order; and so on indefinitely.

Note on the Projection of the General Locus of space of four dimensions into space of three dimensions.

BY THOMAS CRAIG, *Fellow of the Johns Hopkins University.*

IN the first number of this Journal, Professor Newcomb has given a proof of a certain property of a closed surface, viz: that such a surface, if placed in space of four dimensions, could be turned inside out without stretching or tearing it. Of course this is but one out of the many new degrees of freedom which material objects would possess if placed in four-dimensional space; but it is not the object of this brief paper to study the new properties of natural objects placed in space of four dimensions, but rather to determine the representations in space of three dimensions of the loci which can only exist in four-dimensional space. The general locus which characterizes space of four dimensions may be represented by an equation between the four rectangular coordinates x, y, z, t. This locus is, in fact, *the general locus of three dimensions.* A *surface* in four-dimensional space requires for its determination two equations between the variables, and, if the surface is one which may exist in Euclidean space, one of these equations must be linear or of the form

$$\alpha x + \beta y + \gamma z + \delta t + \varepsilon = 0,$$

which denotes the infinite plane Euclidean space. There are of course an infinite number of these Euclidean spaces in four-dimensional space which are determined by assigning different values to the constants α, β, &c.

The two equations which taken together denote a surface in four-dimensional space sustain to this surface the same relation that two equations in Euclidean space sustain to the curve which is the intersection of surfaces corresponding to each of these equations. A surface in four-dimensional space is then given by

$$F(x, y, z, t) = 0,$$
$$\Phi(x, y, z, t) = 0,$$

and $F = 0$ and $\Phi = 0$, each denotes the general three-dimensional locus characteristic of space of four dimensions, just as the surface $F(x, y, z) = 0$ is the general two-dimensional locus characteristic of space of three dimensions.

I obtain in the following paper merely the general differential equations which are necessary to obtain the representation in Euclidean space of the

locus $F(x, y, z, t) = 0$. A transformation to three independent variable parameters would probably have the effect of simplifying the results, as in Euclidean space the whole theory of surfaces, or more exactly of the curvature of surfaces, is more conveniently and elegantly studied by using two independent variable parameters instead of the three dependent rectangular coordinates of any point.

The locus in three-dimensional space $F(x, y, z) = 0$ is projected into two-dimensional space, say the plane $z' = 0$, by simply finding the values of x', y', the rectangular coordinates of the point in the plane corresponding to the point x, y, z of the surface.

Of course the process of finding these values is dependent upon the conditions which are to be fulfilled in making the projection. The condition of projection that we shall here employ is that of the similarity of the elements of the given locus with the representation of these elements in three-dimensional space. The conditions to be fulfilled then are, the preservation of angles in their true size, and the preservation of the ratio between any two elements of the given locus in the projection of these elements.

The equation of the general four-dimensional locus is

$$F(x, y, z, t) = 0.$$

Let ξ, η, ζ denote the rectangular coordinates in space of three dimensions; it is required to find ξ, η, ζ in terms of x, y, z, t, so that the above conditions shall be fulfilled. Since ξ, η, ζ are functions of x, y, z, t, we have

$$d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial z}dz + \frac{\partial \xi}{\partial t}dt,$$
$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz + \frac{\partial \eta}{\partial t}dt,$$
$$d\zeta = \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial y}dy + \frac{\partial \zeta}{\partial z}dz + \frac{\partial \zeta}{\partial t}dt,$$

and also

$$0 = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial t}dt.$$

Write for brevity

$$\frac{\partial F}{\partial x} = \alpha, \quad \frac{\partial F}{\partial y} = \beta, \quad \frac{\partial F}{\partial z} = \gamma, \quad \frac{\partial F}{\partial t} = \varepsilon,$$
$$\frac{\partial \xi}{\partial x} = a, \quad \frac{\partial \xi}{\partial y} = b, \quad \frac{\partial \xi}{\partial z} = c, \quad \frac{\partial \xi}{\partial t} = e,$$
$$\frac{\partial \eta}{\partial x} = a', \quad \frac{\partial \eta}{\partial y} = b', \quad \frac{\partial \eta}{\partial z} = c', \quad \frac{\partial \eta}{\partial t} = e',$$
$$\frac{\partial \zeta}{\partial x} = a'', \quad \frac{\partial \zeta}{\partial y} = b'', \quad \frac{\partial \zeta}{\partial z} = c'', \quad \frac{\partial \zeta}{\partial t} = e'';$$

then the preceding equations become

$$\begin{aligned}\alpha dx + \beta dy + \gamma dz + \varepsilon dt &= 0,\\ adx + bdy + cdz + edt &= d\xi,\\ a'dx + b'dy + c'dz + e'dt &= d\eta,\\ a''dx + b''dy + c''dz + e''dt &= d\zeta.\end{aligned}$$

We will define here ∇ as the determinant of the above equations when $d\xi$, $d\eta$, $d\zeta = 0$, that is,

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \gamma, & \varepsilon \\ a, & b, & c, & e \\ a', & b', & c', & e' \\ a'', & b'', & c'', & e'' \end{vmatrix}.$$

The minors of ∇ corresponding to α, β, γ, ε are

$$K, \; L, \; M, \; N$$

corresponding to a, b, c, e,

$$k, \; l, \; m, \; n$$

and similarly those corresponding to a', b', &c., a'', b'', &c., are

$$k', \; l', \; m', \; n',$$
$$k'', \; l'', \; m'', \; n''.$$

If we suppose, now, that in threefold space ζ is the only quantity which varies, we must have $d\xi = d\eta = 0$, and

$$\begin{aligned}\alpha dx + \beta dy + \gamma dz + \varepsilon dt &= 0,\\ adx + bdy + cdz + edt &= 0,\\ a'dx + b'dy + c'dz + e'dt &= 0;\end{aligned}$$

from these follow

$$\frac{dx}{k''} = -\frac{dy}{l''} = \frac{dz}{m''} = -\frac{dt}{n''}.$$

Write

$$\begin{aligned}\Omega &= \sqrt{k^2 + l^2 + m^2 + n^2},\\ \Omega' &= \sqrt{k'^2 + l'^2 + m'^2 + n'^2},\\ \Omega'' &= \sqrt{k''^2 + l''^2 + m''^2 + n''^2},\end{aligned}$$

then we have for the "direction cosines" of the line on $F = 0$ corresponding to $d\zeta$

$$\frac{k''}{\Omega''}, \; -\frac{l''}{\Omega''}, \; \frac{m''}{\Omega''}, \; -\frac{n''}{\Omega''}.$$

In like manner if η alone vary we have, for the "direction cosines" of the line on F corresponding to $d\eta$,

$$\frac{k'}{\Omega'}, \; -\frac{l'}{\Omega'}, \; \frac{m'}{\Omega'}, \; -\frac{n'}{\Omega'},$$

and, finally, for ξ the only varying quantity

$$\frac{k}{\Omega}, \quad -\frac{l}{\Omega}, \quad \frac{m}{\Omega}, \quad -\frac{n}{\Omega}.$$

The conditions for orthogonality are now

$$kk' + ll' + mm' + nn' = 0,$$
$$k'k'' + l'l'' + m'm'' + n'n'' = 0,$$
$$k''k + l''l + m''m + n''n = 0.$$

The ratio of the element $d\xi$ to the corresponding element on F is

$$\frac{d\xi}{\sqrt{dx^2 + dy^2 + dz^2 + dt^2}},$$

or,

$$\frac{adx + bdy + cdz + edt}{\sqrt{dx^2 + dy^2 + dz^2 + dt^2}},$$

or finally,

$$\frac{ak - bl + cm - en}{\Omega}.$$

Equating this to the ratio of $d\eta$ to its corresponding element on F,

$$\frac{1}{\Omega}(ak - bl + cm - en) = \frac{1}{\Omega'}(a'k' - b'l' + c'm' - e'n').$$

The quantities in the parentheses are respectively $= \nabla$ and $- \nabla$, so that this equation becomes simply

$$\frac{1}{\Omega} - \frac{1}{\Omega'} = 0,$$

or,

$$\Omega' - \Omega = 0,$$

and in like manner

$$\Omega'' - \Omega' = 0,$$
$$\Omega - \Omega'' = 0.$$

These equations, written out in full, constitute, with the previously given equations of orthogonality, the entire system of equations of condition from which we must endeavor to determine the differential equations affording the solution of the problem. For convenience I give the entire set of equations here; they are:

1.) $kk' + ll' + mm' + nn' = 0,$

2.) $k'k'' + l'l'' + m'm'' + n'n'' = 0,$

3.) $k''k + l''l + m''m + n''n = 0,$

4.) $k'^2 + l'^2 + m'^2 + n'^2 - (k^2 + l^2 + m^2 + n^2) = 0,$

5.) $k''^2 + l''^2 + m''^2 + n''^2 - (k'^2 + l'^2 + m'^2 + n'^2) = 0,$

6.) $k^2 + l^2 + m^2 + n^2 - (k''^2 + l''^2 + m''^2 + n''^2) = 0.$

Multiply the first, second and third of these equations by $2i$ and add and substract the results from 4, 5 and 6 respectively.

7. $(k' \pm ik)^2 + (l' \pm il)^2 + (m' \pm im)^2 + (n' \pm in)^2 = 0,$
8. $(k'' \pm ik')^2 + (l'' \pm il')^2 + (m'' \pm im')^2 + (n'' \pm in')^2 = 0,$
9. $(k \pm ik'')^2 + (l \pm il'')^2 + (m \pm im'')^2 + (n \pm in'')^2 = 0.$

The formation of these expressions from the determinant ∇ is not difficult though a little tedious, so I merely give the results in which, however, for brevity, I have written

$$u_1 = \xi + i\eta, \quad v_1 = \eta + i\zeta, \quad w_1 = \zeta + i\xi,$$
$$u_2 = \xi - i\eta, \quad v_2 = \eta - i\zeta, \quad w_2 = \zeta - i\xi,$$

and also used the symbols $u_{1,2}$, $v_{1,2}$, $w_{1,2}$, when no confusion could arise, to denote that the expressions are the same in form for u_2 as for u_1, &c. Equation 7 becomes now

$$\begin{vmatrix} \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t} \\ \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t} \\ \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t}, & \frac{\partial F}{\partial x} \\ \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t}, & \frac{\partial u_{1,2}}{\partial x} \\ \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t}, & \frac{\partial \zeta}{\partial x} \end{vmatrix}^2$$
$$+ \begin{vmatrix} \frac{\partial F}{\partial t}, & \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \\ \frac{\partial u_{1,2}}{\partial t}, & \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y} \\ \frac{\partial \zeta}{\partial t}, & \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z} \\ \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z} \\ \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z} \end{vmatrix}^2 = 0.$$

The expanded forms of 8 and 9 are obtained from this by merely changing u into v and w successively. Denoting by θ_1, θ_2, θ_3, θ_4 any four quantities whatever and writing,

$$\begin{vmatrix} \theta_1, & \theta_2, & \theta_3, & \theta_4 \\ \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t} \\ \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t} \\ \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t} \end{vmatrix} = U_{1,2},$$

we can place the last equation in the form

$$\left(\frac{\partial U_{1,2}}{\partial \theta_1}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_2}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_3}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_4}\right)^2 = 0,$$

or, for convenience, making

$$\left(\frac{\partial}{\partial\theta_1}\right)^2+\left(\frac{\partial}{\partial\theta_2}\right)^2+\left(\frac{\partial}{\partial\theta_3}\right)^2+\left(\frac{\partial}{\partial\theta_4}\right)^2\equiv 0,$$

$\left(\text{observe that } \left(\frac{\partial}{\partial\theta_i}\right)^2 \text{ is not } = \frac{\partial^2}{\partial\theta_i^2}\right)$,

$$DU_{1,2}=0,\quad DV_{1,2}=0,\quad DW_{1,2}=0,$$

V and W being quantities of the same kind as U.

The calculation of the quantities $\left(\frac{\partial U_{1,2}}{\partial\theta_i}\right)^2$ is quite simple, since each of them is a symmetrical determinant. If we expand these expressions so that $DU_{1,2}$ shall appear in the form

$$A\alpha^2+B\beta^2+C\gamma^2+D\varepsilon^2-2E\alpha\beta-2F\beta\gamma, \&c.=0,$$

$\alpha, \beta, \gamma, \varepsilon$ being as before the derivatives of F with respect to x, y, z, t, we shall find A, B, C, D given as sums of squares of three determinants of the second order—or, consequently, as one symmetrical determinant of that order; E, F, &c. as sums of products of determinants of the second order. But the most convenient form in which to have the equation is that of one symmetrical determinant of the third order, and by a known theorem we can write it at once as

$$DU_{1,2}=\begin{vmatrix} \alpha^2+\beta^2+\gamma^2+\varepsilon^2, & a_1\alpha+b_1\beta+c_1\gamma+e_1\varepsilon, & a_2\alpha+b_2\beta+c_2\gamma+e_2\varepsilon \\ a_1\alpha+b_1\beta+c_1\gamma+e_1\varepsilon, & a_1^2+b_1^2+c_1^2+e_1^2, & a_1a_2+b_1b_2+c_1c_2+e_1e_2 \\ a_2\alpha+b_2\beta+c_2\gamma+e_2\varepsilon, & a_1a_2+b_1b_2+c_1c_2+e_1e_2, & a_2^2+b_2^2+c_2^2+e_2^2 \end{vmatrix}=0,$$

where a_1, b_1, c_1, e_1 are the derivatives of $u_{1,2}$ with respect to x, y, z, t respectively, a_2, b_2, c_2, e_2 are the same derivatives of ζ. There are two other equations of this form for $v_{1,2}$ and $w_{1,2}$ in which $\alpha, \beta, \gamma, \varepsilon$ have the same significance as in $DU_{1,2}=0$, a_1, b_1, c_1, e_1 are the derivatives however of $v_{1,2}$ and $w_{1,2}$, respectively, and a_2, b_2, c_2, e_2 denote in $DV_{1,2}=0$ the derivatives of ξ and in $DW_{1,2}=0$ the derivatives of η with respect to x, y, z and t.

If we had started to project the surface $F(x, y, z)=0$ in Euclidean space upon the plane ξ, η, we would have found as the equations of the problem

$$\begin{vmatrix} \alpha^2+\beta^2+\gamma^2, & a_1\alpha+b_1\beta+c_1\gamma \\ a_1\alpha+b_1\beta+c_1\gamma, & a_1^2+b_1^2+c_1^2 \end{vmatrix}=0,$$

the α, β, γ having the same meaning, as before, and a_1, b_1, c_1 to take the values

$$\frac{\partial u_1}{\partial x},\quad \frac{\partial u_1}{\partial y},\quad \frac{\partial u_1}{\partial z},$$
$$\frac{\partial u_2}{\partial x},\quad \frac{\partial u_2}{\partial y},\quad \frac{\partial u_2}{\partial z},$$

successively. These are the equations for the orthomorphic projection of a surface upon a plane.

If we have n variables $x_1, x_2, \ldots x_n$ connected by the relation

$$F(x_1, x_2, \ldots x_n) = 0,$$

the general locus of n-dimensional space, and $n-1$ variables $\xi_1, \xi_2, \ldots \xi_{n-1}$ denoting the coordinates in space of $n-1$ dimensions, then for the projection of $F=0$ into the $n-1$-dimensional space, we shall have $\frac{(n-1)(n-2)}{2}$ groups of equations of the form

$$\text{A.} \qquad \left\{ \begin{aligned} &\left(\frac{\partial U_{1,2}}{\partial\theta_1}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial\theta_2}\right)^2 + \ldots \left(\frac{\partial U_{1,2}}{\partial\theta_n}\right)^2 = 0, \\ &\left(\frac{\partial V_{1,2}}{\partial\theta_1}\right)^2 + \left(\frac{\partial V_{1,2}}{\partial\theta_2}\right)^2 + \ldots \left(\frac{\partial V_{1,2}}{\partial\theta_n}\right)^2 = 0, \end{aligned} \right.$$

the quantities θ being anything whatever, and the derivatives

$$\frac{\partial U}{\partial\theta}, \quad \frac{\partial V}{\partial\theta},$$

being the first minors of the determinants

$$\begin{vmatrix} \theta_1, & \theta_2, & \ldots & \theta_n \\ \frac{\partial F}{\partial x_1}, & \frac{\partial F}{\partial x_2}, & \ldots & \frac{\partial F}{\partial x_n} \\ \frac{\partial u_{1,2}}{\partial x_1}, & \frac{\partial u_{1,2}}{\partial x_2}, & \ldots & \frac{\partial u_{1,2}}{\partial x_n} \\ \frac{\partial \xi_3}{\partial x_1}, & \frac{\partial \xi_3}{\partial x_2}, & \ldots & \frac{\partial \xi_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \xi_{n-1}}{\partial x_1}, & \frac{\partial \xi_{n-1}}{\partial x_1}, & \ldots & \frac{\partial \xi_{n-1}}{\partial x_n} \end{vmatrix} = V_{1,2}$$

and

$$\begin{vmatrix} \theta_1, & \theta_2, & \ldots & \theta_n \\ \frac{\partial F}{\partial x_1}, & \frac{\partial F}{\partial x_2}, & \ldots & \frac{\partial F}{\partial x_n} \\ \frac{\partial v_{1,2}}{\partial x_1}, & \frac{\partial v_{1,2}}{\partial x_2}, & \ldots & \frac{\partial v_{1,2}}{\partial x_n} \\ \frac{\partial \xi_3}{\partial x_1}, & \frac{\partial \xi_3}{\partial x_2}, & \ldots & \frac{\partial \xi_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \xi_{n-1}}{\partial x_1}, & \frac{\partial \xi_{n-1}}{\partial x_2}, & \ldots & \frac{\partial \xi_{n-1}}{\partial x_n} \end{vmatrix} = V_{1,2}$$

where

$$u_{1,2} = \xi_1 + i\xi_2, \quad u_{2,3} = \xi_2 + i\xi_3, \; \ldots \; u_{i,i+1} = \xi_i + i\xi_{i+1},$$
$$v_{1,2} = \xi_1 - i\xi_2, \quad v_{2,3} = \xi_2 - i\xi_3, \; \ldots \; v_{i,i+1} = \xi_i - i\xi_{i+1}.$$

The remaining $\frac{n(n-3)}{2}$ groups of equations similar to A are of course formed from this group by changing the $u_{1,2}$ into $u_{2,3}$, $u_{3,4}$, &c., and $v_{1,2}$ into $v_{2,3}$, $v_{3,4}$, &c., and also changing the ξ by advancing successively all the subscripts $1, 2 \ldots n-1$.

By a theorem of determinants (which was unknown to me until kindly communicated by Mr. Stringham) each of equations A may be put in the following form:

$$\begin{vmatrix} \Sigma\left(\frac{\partial F}{\partial x_i}\right)^2, & \Sigma\frac{\partial F}{\partial x_i}\frac{\partial u_{1,2}}{\partial x_i}, & \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & & \ldots\ldots\ldots\ldots\Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\ \Sigma\frac{\partial F}{\partial x_i}\frac{\partial u_{1,2}}{\partial x_i}, & \Sigma\left(\frac{\partial u_{1,2}}{\partial x_i}\right)^2, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & & \ldots\ldots\ldots\ldots\Sigma\frac{\partial u_{1,3}}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\ \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \Sigma\left(\frac{\partial \xi_3}{\partial x_i}\right)^2, & & \ldots\ldots\ldots\ldots\Sigma\frac{\partial \xi_3}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\ \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\frac{\partial \xi_3}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\left(\frac{\partial \xi_4}{\partial x_i}\right)^2, & \ldots\Sigma\frac{\partial \xi_4}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i}, & \ldots\ldots & \ldots\ldots & \ldots\ldots\Sigma\left(\frac{\partial \xi_{n-1}}{\partial x_i}\right)^2 \end{vmatrix} = 0,$$

the summations to be taken from $i = 1$ to $i = n$. The equation for V being similar to this, there is no necessity for writing it.

WASHINGTON, *Sept. 4, 1879.*

On the Motion of an Ellipsoid in a Fluid.

BY THOMAS CRAIG, *Fellow of the Johns Hopkins University.*

LET us suppose a solid body of the form of an ellipsoid whose mass is so distributed as to be symmetrical with respect to the three principal planes of the body; that this body be immersed in an infinite mass of frictionless incompressible fluid, the whole system being originally at rest, and that the solid be acted upon by any set of impulsive forces—or by any *impulse*—and the system then left to itself; to determine the resulting motion of the solid body and of the fluid. Concerning the motion of the fluid we can say that it will be subject to a velocity potential, since the entire motion of the fluid being due to that of the solid—or to the motion of a portion of the bounding surface of the fluid—there can be no rotation in any part of the mass.

We will now proceed to the determination of the velocity potential ϕ satisfying the equation

$$\nabla^2\phi = 0,$$

∇^2 as usual standing for the operation

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

As we limit ourselves to a simply connected space, the function ϕ will be single valued.

Let now ξ, η, ζ denote the coordinates of a point with respect to a set of rectangular axes fixed in space, and also let x, y, z denote the coordinates of a point with respect to a similar set of axes fixed in the body, we have then

$$\begin{aligned}
\xi &= \alpha + \alpha_1 x + \alpha_2 y + \alpha_3 z, \\
\eta &= \beta + \beta_1 x + \beta_2 y + \beta_3 z, \\
\zeta &= \gamma + \gamma_1 x + \gamma_2 y + \gamma_3 z,
\end{aligned}$$

the twelve quantities α, β, γ being functions of the time and position of the body whose geometric meaning is well known.

If we denote by u, v, w the translation velocities of the origin of x, y, z in these respective directions and by p, q, r the components of angular velo-

city around these same axes, we have for the velocity of the fluid particles relatively to the body

$$
\begin{aligned}
&\frac{dx}{dt} = \frac{\partial\varphi}{\partial x} - u - zq + yr = \mathrm{U}\,, \\
1. \qquad &\frac{dy}{dt} = \frac{\partial\varphi}{\partial y} - v - xr + zp = \mathrm{V}\,, \\
&\frac{dz}{dt} = \frac{\partial\varphi}{\partial z} - w - yp + xq = \mathrm{W}\,.
\end{aligned}
$$

If we denote by (nx), (ny), (nz) the angles which the outer normal to the surface of the body makes with the axes of x, y, z respectively, we have

$$\frac{\partial\varphi}{\partial n} = (u + zq - yr)\cos(nx) + (v + xr - zp)\cos(ny) + (w + yp - xq)\cos(nz)\,.$$

As the fluid is to be at rest at an infinite distance from the body, the first derivatives of ϕ with respect to x, y, z vanish at infinity; and since $\nabla^2\phi = 0$ throughout the space filled by the fluid, and the quantities $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$, $\frac{\partial\varphi}{\partial z}$ are continuous throughout the same, we have (Kirchhoff, pg. 225) for ϕ the expression

$$\phi = u\phi_1 + v\phi_2 + w\phi_3 + p\phi_4 + q\phi_5 + r\phi_6\,.$$

The six functions ϕ_1, ϕ_2, ... ϕ_6 satisfying the equation $\nabla^2\phi = 0$ and also giving at the surface of the body

$$
\begin{aligned}
&\frac{\partial\varphi_1}{\partial u} = \cos(nx)\,, \quad &&\frac{\partial\varphi_4}{\partial u} = y\cos(uz) - z\cos(ny)\,, \\
2. \qquad &\frac{\partial\varphi_2}{\partial u} = \cos(ny)\,, \quad &&\frac{\partial\varphi_5}{\partial u} = z\cos(ux) - x\cos(nz)\,, \\
&\frac{\partial\varphi_3}{\partial u} = \cos(nz)\,, \quad &&\frac{\partial\varphi_6}{\partial u} = x\cos(uy) - y\cos(nx)\,,
\end{aligned}
$$

From these equations we see that the functions ϕ_1, ϕ_2, ... depend simply on the form of the surface of the body, and not at all on its motion, ϕ being a linear function of the quantities u, v, ... r.

We shall resume now the examination of the function ϕ, observing that, in addition to the equations already given to be satisfied at the surface of the body and throughout the space filled by the fluid, it must also satisfy for $\lambda_1 = \infty$ the equation

$$3. \qquad \left[\frac{1}{E^2}\left(\frac{\partial\varphi}{\partial\lambda_1}\right) + \frac{1}{F^2}\left(\frac{\partial^2\varphi^2}{\partial\lambda_2}\right) + \frac{1}{G^2}\left(\frac{\partial\varphi^2}{\partial\lambda_3}\right)\right] = 0.$$

Denoting by Ω the potential of the ellipsoid upon any point x, y, z, we have for a point at the surface of the body

$$\Omega = \text{const.} - \pi\,(Ax^2 + By^2 + Cz^2)$$

(Kirchhoff, pg. 130), where

4. $$A = abc\int\frac{d\lambda_1}{(a^2+\lambda_1)N},\quad B = abc\int\frac{d\lambda_1}{(b^2+\lambda_1)N},\quad C = abc\int\frac{d\lambda_1}{(c^2+\lambda_1)N}$$
$$N = \sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)},$$

and for the derivatives of Ω (ib. pg. 225.)

5. $$\begin{aligned}\frac{\partial^2\Omega}{\partial n\partial x} &= 2\pi(2-A)\cos(nx),\\ \frac{\partial^2\Omega}{\partial n\partial y} &= 2\pi(2-B)\cos(ny),\\ \frac{\partial^2\Omega}{\partial n\partial z} &= 2\pi(2-C)\cos(nz).\end{aligned}$$

Comparing these expressions with the values previously obtained for $\frac{\partial\varphi_1}{\partial n}$ &c., we have

6. $$\phi_1 = \frac{1}{2\pi(2-A)}\frac{\partial\Omega}{\partial x},\quad \phi_2 = \frac{1}{2\pi(2-B)}\frac{\partial\Omega}{\partial y},\quad \phi_3 = \frac{1}{2\pi(2-C)}\frac{\partial\Omega}{\partial z}.$$

All the conditions that ϕ_6 must satisfy with the exception of

$$\frac{\partial\varphi_6}{\partial n} = x\cos(ny) - y\cos(nx)$$

will be satisfied if we write

7. $$\phi_6 = N_3\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right)$$

when N_3 is an arbitrary constant; Kirchhoff shows that this condition will also be satisfied by a proper determination of N_3; the value which he obtains in a very simple manner is

8. $$N_3 = \frac{a^2-b^2}{2\pi[2(a^2-b^2)+(A-B)(a^2+b^2)]},\ \&c.$$

If instead of N_1, N_2, N_3 equal to the values here given we write $2\pi N_1$, &c., it will be a little more convenient, and we now have for ϕ

9. $$\begin{aligned}\phi = {} & \frac{Au}{A-2}x + \frac{Bv}{B-2}y + \frac{Cw}{C-2}z\\ & + N_1(B-C)yzp + N_2(C-A)zxq + N_3(A-B)xyr.\end{aligned}$$

The quantities A, B, C are given by the expressions

$$\begin{aligned}A &= -bc\frac{\partial\omega_1}{\partial a},\\ B &= -ca\frac{\partial\omega_1}{\partial b},\\ C &= -ab\frac{\partial\omega_1}{\partial c},\end{aligned}$$

where $$\omega_1 = \int \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$
and for convenience of future reference
$$\omega_2 = \int \frac{d\lambda_2}{\sqrt{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}},$$
$$\omega_3 = \int \frac{d\lambda_3}{\sqrt{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)}}.$$

We can arrive at the final form of ϕ in a slightly different manner, which will bring in evidence some interesting properties of the quantities that we are dealing with. For simplicity, suppose that the ellipsoid moves with a constant velocity u in the direction of x, then, as before,
$$\phi = u\phi_1,$$
with the prescribed conditions for ϕ_1, will satisfy all the conditions of the motion, and will give again
$$\phi_1 = \frac{1}{2\pi(2-A)}\frac{\partial\Omega}{\partial x}.$$
Write $$A_1,\ B_1,\ C_1 = \frac{A}{abc},\ \frac{B}{abc},\ \frac{C}{abc},$$
Now $$A+B+C = abc\left\{\int_0^\infty \frac{d\lambda_1}{(a^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right.$$
$$\left.+\int_0^\infty \frac{d\lambda_1}{(b^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}} + \int_0^\infty \frac{d\lambda_1}{(c^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right\}$$
$$= abc\int_0^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\left[\frac{(b^2+\lambda_1)(c^2+\lambda_1)+(a^2+\lambda_1)(c^2+\lambda_1)+(a^2+\lambda_1)(b^2+\lambda_1)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\right]$$
$$= \frac{2\,.\,abc}{abc},$$
or $$A_1+B_1+C_1 = \frac{2}{abc},$$
now

10. $$\phi_1 = \frac{1}{2\pi(2-A)}\frac{\partial\Omega}{\partial x} = -\frac{abc}{(2-A)}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}},$$
where χ is the positive root of
$$\frac{x^2}{a^2+\chi}+\frac{y^2}{b^2+\chi}+\frac{z^2}{c^2+\chi}=1;$$
substituting for A its value, we have again
$$\phi_1 = \frac{\displaystyle\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}}{\displaystyle\int_0^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}-\frac{2}{abc}} = \frac{1}{B_1+C_1}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}$$

and consequently

11. $$\phi = \frac{u}{B_1 + C_1} \int_\chi^\infty \frac{x d\lambda_1}{\sqrt{(a^2+\lambda_1)^3 (b^2+\lambda_1)(c^2+\lambda_1)}}.$$

From this value of ϕ we can readily find for U, V, W the values

12.
$$\mathrm{U} = \frac{u}{B_1 + C_1} \left\{ \int_\chi^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} - \frac{x}{\sqrt{(a^2+\chi_1)^3(b^2+\chi_1)(c^2+\chi_1)}} \frac{2x}{a^2+\chi} \times \left[\frac{x^2}{(a^2+\chi)^2} + \frac{y^2}{(b^2+\chi)^2} + \frac{z^2}{(c^2+\chi)^2}\right]^{-1} \right\},$$
$$\mathrm{V} = -\frac{u}{B_1+C_1} \frac{x}{\sqrt{(a^2+\chi)^3(b^2+\chi)(c^2+\chi)}} \cdot \frac{2z}{b^2+\chi} \cdot \left[\frac{x^2}{(a^2+\chi)^2} + \frac{y^2}{(b^2+\chi)^2} + \frac{z^2}{(c^2+\chi)^2}\right]^{-1},$$
$$\mathrm{W} = -\frac{u}{B_1+C_1} \frac{x}{\sqrt{(a^2+\chi)^3(b^2+\chi)(c^2+\chi)}} \cdot \frac{2y}{c^2+\chi} \cdot \left[\frac{x^2}{(a^2+\chi)^2} + \frac{y^2}{(b^2+\chi)^2} + \frac{z^2}{(c^2+\chi)^2}\right]^{-1},$$

since $$\frac{\partial \chi}{\partial x} = \frac{2x}{a^2+\chi} \left[\frac{x^2}{(a^2+\chi)^2} + \frac{y^2}{(b^2+\chi)^2} + \frac{z^2}{(c^2+\chi)^2}\right]^{-1}, \text{ \&c., \&c.}$$

If we suppose now that
$$\phi = u\phi_1 + v\phi_2 + w\phi_3,$$
i. e. that the body move in any direction with the component velocities u, v, w, we see that all we have to do is to determine separately ϕ_2 and ϕ_3 and, substituting in this expression for ϕ, have the motion completely determined. Obviously the functions ϕ_2 and ϕ_3 are given by
$$\phi_2 = \frac{1}{A_1 + C_1} \int_\chi^\infty \frac{y d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}},$$
$$\phi_3 = \frac{1}{A_1 + B_1} \int_\chi^\infty \frac{z d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}},$$
and thus for a simple motion of translation we have

14. $$\phi = \frac{u}{B_1 + C_1} \int_\chi^\infty \frac{x d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} + \frac{v}{C_1 + A_1} \int_\chi^\infty \frac{y d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}} + \frac{w}{A_1 + B_1} \int_\chi^\infty \frac{z da_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}}.$$

Similarly in the case where there is rotation we find the values

15.
$$N_1 = \frac{1}{2\pi} \cdot \frac{1}{\left(\frac{2}{abcA_2} - (b^2+c^2)\right)}, \quad A_2 = \int_0^\infty \frac{da_1}{(c^2+\lambda_1)B_1},$$
$$N_2 = \frac{1}{2\pi} \cdot \frac{1}{\left(\frac{2}{abcB_2} - (c^2+a^2)\right)}, \quad B_2 = \int_0^\infty \frac{da_1}{(a^2+\lambda_1)C_1},$$
$$N_3 = \frac{1}{2\pi} \cdot \frac{1}{\left(\frac{2}{abcC_2} - (a^2+b^2)\right)}, \quad C_2 = \int_0^\infty \frac{da_1}{(b^2+\lambda_1)A_1}.$$

So that finally we can write for the general value of ϕ the following

16.
$$\phi = \frac{u}{B_1 + C_1}\int_x^\infty \frac{xd\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} + \frac{v}{A_1 + C_1}\int_x^\infty \frac{yd\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}}$$
$$+ \frac{w}{A_1 + B_1}\int_x^\infty \frac{zd\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}}$$
$$+ \frac{(b^2 - c^2)\,p}{(b^2 + c^2)\,A_2 - \dfrac{2}{abc}}\int_x^\infty \frac{yzd\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)^3}}$$
$$+ \frac{(c^2 - a^2)\,q}{(c^2 + a^2)\,B_2 - \dfrac{2}{abc}}\int_x^\infty \frac{zxd\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)^3}}$$
$$+ \frac{(a^2 - b^2)\,r}{(a^2 + b^2)\,C_2 - \dfrac{2}{abc}}\int_x^\infty \frac{xyd\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)^3(c^2+\lambda_1)}}\,.$$

The approximate values of the quantities A, B, C can be obtained by expanding ω_1 in an infinite series and then performing the indicated differentiations, which is the method employed by Clebsch for determining their values; we can also obtain them directly as elliptic functions capable all of being reduced to depend upon the ϑ-function. If we take χ_1, χ_2, χ_3 as the amplitudes of three elliptic integrals

17.
$$\theta_1 = \int\frac{d\chi_1}{\varDelta(k, \chi_1)}, \quad \theta_2 = \int\frac{d\chi_2}{\varDelta(k, \chi_2)}, \quad \theta_3 = \int\frac{d\chi_3}{\varDelta(k, \chi_3)}$$

we shall be able to give A, B, C as elliptic functions of any one of these three quantities θ_1, θ_2, θ_3 and could determine ϕ as a function of all three, *i. e.* supposing the proper relations to exist between these quantities and the parameters λ_1, λ_2, λ_3; these relations must evidently be such as to determine θ_1, θ_2, θ_3 as the so-called Lamé or ellipsoidal coordinates. First to determine A, B and C, write

18.
$$\lambda_1 = c^2\frac{h^2 - x^2}{x^2}, \quad \text{when } h^2 = \frac{a^2 - c^2}{c^2}$$

x having all values between zero and infinity. This gives us

19.
$$a^2 + \lambda_1 = (a^2 - c^2)\,\frac{x^2 + 1}{x^2},$$
$$b^2 + \lambda_1 = \frac{(b^2 - c^2)\,x^2 + (a^2 - c^2)}{x^2},$$
$$c^2 + \lambda_1 = \frac{a^2 - c^2}{x^2},$$
$$d\lambda_1 = -\frac{2\,(a^2 - c^2)}{x^3}\,dx\,.$$

Assume $\frac{a^2-b^2}{a^2-c^2}=k^2$ and, consequently, for the complementary modulus $\frac{b^2-c^2}{a^2-c^2}=k'^2$; making these substitutions and, for brevity, writing $I=\frac{2abc}{(a^2-c^2)^{\frac{3}{2}}}$, we have

$$A=I\int_0^\infty \frac{x^2dx}{(x^2+1)^{\frac{3}{2}}\sqrt{1+k'^2x^2}},$$

20. $$B=I\int_0^\infty \frac{x^2dx}{(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}},$$

$$C=I\int_0^\infty \frac{x^2dx}{\sqrt{(x^2+1)(1+k'^2x^2)}}.$$

Again, writing $x=\tan\chi_1$, where χ_1 lies between 0 and $\frac{\pi}{2}$, we have readily

$$A=I\int_0^{\frac{\pi}{2}} \frac{\sin^2\chi_1 d\chi_1}{\varDelta(k,\chi_1)}=I\int_0^{\frac{\pi}{2}} \mathrm{sn}^2\,\theta_1 d\theta_1,$$

21. $$B=\frac{I}{k^2}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\varDelta^2(k,\chi_1)}-1\right)\frac{d\chi_1}{\varDelta(k,\chi_1)}=I\int_0^{\frac{\pi}{2}}\left(\frac{1}{\mathrm{dn}^2\theta_1}-1\right)d\theta_1,$$

$$C=I\int_0^{\frac{\pi}{2}}\frac{\sin^2\chi_1}{\cos^2\chi_1}\,\frac{d\chi_1}{\varDelta(k,\chi_1)}=I\int_0^{\frac{\pi}{2}}\frac{\mathrm{sn}^2\,\theta_1}{\mathrm{cn}^2\,\theta_1}\,d\theta_1.$$

Now we know (Cayley, Elliptic Func., p. 15) that

$$k^2\int \mathrm{sn}^2\,\theta_1 d\theta_1=\left(1-\frac{E}{K}\right)\theta_1-Z\theta_1,$$

and so we have readily for A_1

22. $$A=\frac{I}{k^2}\left\{\left(1-\frac{E}{K}\right)\theta_1-Z\theta_1\right\}$$
$$=-\frac{I}{k^2}\left\{\frac{k\theta_1}{K}\frac{\partial E}{\partial k}+\frac{\theta'(\theta_1)}{\theta(\theta_1)}\right\},$$

and also by not difficult reductions

$$B=\frac{I}{k^2}\left\{\left(\frac{E}{k'^2K}-1\right)\theta_1+\frac{1}{k'^2}Z(\theta_1+K)\right\}$$

22. $$=\frac{I}{k^2}\left\{\frac{\theta_1}{k}\frac{\partial\log K}{\partial k}+\frac{1}{k'^2}\frac{\theta'(\theta_1+K)}{\theta(\theta_1+K)}\right\},$$

$$C=\frac{I}{k'^2}\left\{\frac{\partial}{\partial\theta_1}\log H(\theta_1+K)-\frac{E}{K}\theta_1\right\}.$$

The expression for C could be given as depending upon the ϑ-function by the formula

$$H(\theta_1+K)=\varepsilon^{-\frac{\pi}{4K}(K'-2i\theta)}\,\Theta(\theta_1+K),$$

but nothing would be gained thereby. We may just notice the forms which immediately present themselves for the constants A_2, B_2 and C_2, but will not

integrate the equations. For brevity, we will here denote $\dfrac{2}{(a^2-c^2)^{\frac{5}{2}}}$ by I', and we have by the same transformations employed in reducing the quantities A, B and C,

$$
\begin{aligned}
A_2 &= I'\int\frac{x^4dx}{(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d_1\chi}{\cos^2\chi_1\Delta^3(k,\chi_1)},\\
B_2 &= I'\int\frac{x^4dx}{(x^2+1)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d\chi_1}{\cos^2\chi_1\Delta(k,\chi_1)},\\
C_2 &= I\int\frac{x^4dx}{(x^2+1)(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d\chi_1}{\Delta^3(k,\chi_1)};
\end{aligned}
\tag{23.}
$$

and these, by virtue of the equation $\chi_1 = \mathrm{am}\theta_1$ become

$$
A_3 = I\int\frac{\mathrm{sn}^2\theta_1}{\mathrm{dn}^2\theta_3}\,\frac{\mathrm{sn}^2\theta_1}{\mathrm{cn}^2\theta_1}\,d\theta_1,\quad B_2 = I\int\mathrm{sn}^2\theta_1\,\frac{\mathrm{sn}^2\theta_1}{\mathrm{cn}^2\theta_1}\,d\theta_1,\quad C_2 = I\int\mathrm{sn}^2\theta_1\,\frac{\mathrm{sn}^2\theta_1}{\mathrm{dn}^2\theta_1}\,d\theta_1.
\tag{24.}
$$

The values of x, y, and z can, of course, be expressed by means of the functions θ_1, θ_2, θ_3. The values of x^2, y^2 and z^2, expressed in terms of the parameters λ_1, λ_2, λ_3, are (see equations 50. in article on "*the Motion of a Solid in a Fluid;*" this Journal, Vol. II, No. 2.)

$$
\begin{aligned}
x^2 &= \frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)},\\
y^2 &= \frac{(b^2+\lambda_1)(b^2+\lambda_2)(b^2+\lambda_3)}{(b^2-c^2)(b^2-a^2)},\\
z^2 &= \frac{(c^2+\lambda_1)(c^2+\lambda_2)(c^2+\lambda_3)}{(a^2-c^2)(b^2-c^2)};
\end{aligned}
\tag{25.}
$$

performing here operations similar to those that have already been employed in transforming to θ_1, we have, since θ_2 and θ_3 sustain the same kind of a relation to λ_2 and λ_3 that θ_1 does to λ_1,

$$
\begin{aligned}
x^2 &= (a^2-c^2)\,\frac{\mathrm{dn}^2\theta_2\,\mathrm{sn}^2\theta_3}{\mathrm{sn}^2\theta_1}, & x &= \sqrt{a^2-c^2}\,\frac{\mathrm{dn}\theta_2\,\mathrm{sn}\theta_3}{\mathrm{sn}\theta_1};\\
y^2 &= (a^2-c^2)\,\frac{\mathrm{dn}^2\theta_1\,\mathrm{cn}^2\theta_2\,\mathrm{cn}^2\theta_3}{\mathrm{sn}^2\theta_1}, & y &= \sqrt{a^2-c^2}\,\frac{\mathrm{dn}\theta_1\,\mathrm{cn}\theta_2\,\mathrm{cn}\theta_3}{\mathrm{sn}\theta_1};\\
z^2 &= (a^2-c^2)\,\frac{\mathrm{cn}^2\theta_1\,\mathrm{sn}^2\theta_2\,\mathrm{dn}^2\theta_3}{\mathrm{sn}^2\theta_1}, & z &= \sqrt{a^2-c^2}\,\frac{\mathrm{cn}\theta_1\,\mathrm{sn}\theta_2\,\mathrm{dn}\theta_3}{\mathrm{sn}\theta_1}.
\end{aligned}
\tag{26.}
$$

With these values of x, y, z, A, B, C, A_2, B_2, C_2, we could now transform ϕ so that it should be given as a function of elliptic functions, but nothing of interest could come from that operation in the general case where

none of the velocities vanish, so we will not attempt it. For the case of $k=0$, or an oblate ellipsoid we have readily

27. $$A=B=\frac{I_0}{2}(\theta_1-\sin\theta_1\cos\theta_1)$$
$$C=I_0(\tan\theta_1-\theta_1).$$

Suppose $k=1$, *i. e.* a prolate ellipsoid, then we have

$$\chi_1=\text{gd}\theta_1$$

and $\Delta(k,\chi_1)=\cos\chi_1=\text{cg}\theta_1$, &c. We have in this case, as is well known,

$$\theta_1=\log\tan\left(\frac{1}{2}\pi+\frac{1}{2}\chi_1\right)=\log\tan\left(\frac{1}{2}\pi+\frac{1}{2}\text{gd}\theta_1\right),$$

and, passing to exponentials and reducing,

$$\varepsilon^{\theta_1}=\frac{\text{cg}\theta_1}{1-\text{sg}\theta_1},$$

and thence

$$\text{sg}\theta_1=\frac{\varepsilon^{\theta_1}-\varepsilon^{-\theta_1}}{\varepsilon^{\theta_1}+\varepsilon^{\theta_1}}=\tanh\theta_1,$$

$$\text{cg}\theta_1=\frac{1}{\varepsilon^{\theta_1}+\varepsilon^{-\theta_1}}\text{sech}\,\theta_1.$$

Substituting in the general values for A, B and C and we have

28. $$A=I\int\text{sg}^2\theta_1\,d\theta_1=I\int(1-\text{cg}^2\theta_1)\,d\theta_1,$$

or,

$$A=I(\theta_1-\text{sg}\theta_1)=I_1(\theta_1-\tanh\theta_1);$$

and similarly,

28'. $$B=C=I_1\int\frac{\text{sg}^2\theta_1}{\text{cg}^2\theta_1}\,d\theta_1=I_1\int\frac{\tanh^2\theta_1}{\text{sech}^2\theta_1}\,d\theta_1$$
$$=I_1\int\sinh^2\theta_1\,d\theta=\frac{I}{4}\left\{\frac{\sinh^2\theta_1}{2}+(1-2\theta_1)\right\},$$

I_1 being what I becomes for $b=c$.

The motion of the fluid particles relatively to the body is given in rectilinear rectangular coordinates by equations 1., but, as for the problem under consideration it is more convenient to use elliptic coordinates, we will consider equations 39. of the article on "*the Motion of a Solid in a Fluid*" above cited.

We will first assume the case where the ellipsoid has merely a motion of translation in the direction of one of its axes—the axis of x for example; for this case we have

$$v=w=p=q=r=0,$$

and the resulting equations of motion,

29.
$$E^2\frac{\partial\lambda_1}{\partial t}=\frac{\partial\varphi}{\partial\lambda_1}-u\frac{\partial x}{\partial\lambda_1},$$
$$F^2\frac{\partial\lambda_2}{\partial t}=\frac{\partial\varphi}{\partial\lambda_2}-u\frac{\partial x}{\partial\lambda_2},$$
$$G^2\frac{\partial\lambda_3}{\partial t}=\frac{\partial\varphi}{\partial\lambda_3}-u\frac{\partial x}{\partial\lambda_3}.$$

These are the equations given and discussed by Clebsch. We have also for ϕ

$$\phi=\frac{A}{A-2}\,xu=\frac{Ax}{A-2}\cdot\frac{\partial a}{\partial t},$$

and

$$\frac{\partial x}{\partial\lambda_1}=\frac{1}{2}\,\frac{x}{a^2+\lambda_1},\quad \frac{\partial x}{\partial\lambda_2}=\frac{1}{2}\,\frac{x}{a^2+\lambda_2},\quad \frac{\partial x}{\partial\lambda_3}=\frac{1}{2}\,\frac{x}{a^2+\lambda_3},$$

these can clearly be expressed simply in terms of the elliptic functions—but nothing would be gained by it, as the resulting equations of motion will give rise to integrals of a higher order than elliptic. Making these substitutions in the differential equations of motion and they become after simple reductions

30.
$$\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\cdot\frac{\partial\lambda_1}{\partial a}=\frac{2x}{a^2+\lambda_1}\cdot\frac{2}{A-2}\left\{1-\frac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right\},$$
$$\frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}\cdot\frac{\partial\lambda_2}{\partial a}=\frac{2x}{a^2+\lambda_2}\cdot\frac{2}{A-2},$$
$$\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)}\cdot\frac{\partial\lambda_3}{\partial a}=\frac{2x}{a^2+\lambda_3}\cdot\frac{2}{A-2}.$$

Making

$$\frac{4x}{A-2}\cdot\frac{da}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}=d\Psi$$

we can write these equations in the simpler form

31.
$$(\lambda_2-\lambda_3)\,d\Psi=\frac{d\lambda_1}{(b^2+\lambda_1)(c^2+\lambda_1)}\cdot\frac{1}{1-\dfrac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}},$$
$$(\lambda_3-\lambda_1)\,d\Psi=\frac{d\lambda_2}{(b^2+\lambda_2)(c^2+\lambda_2)},$$
$$(\lambda_1-\lambda_2)\,d\Psi=\frac{d\lambda_3}{(b^2+\lambda_3)(c^2+\lambda_3)},$$

Multiply these equations respectively by

$$c^2+\lambda_1,\quad c^2+\lambda_2,\quad c_2+\lambda_3,$$

and again by

$$b^2+\lambda_1,\quad b^2+\lambda_2,\quad b^2+\lambda_3,$$

and add each of the sets thus obtained, observing that the quantity

$$\frac{1}{1-\dfrac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}}$$

can be written

$$1-\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$

and we have for the equations of the motion of the fluid particle relatively to the body

32.* $$2\frac{dy}{y}=\frac{d\lambda_1}{b^2+\lambda_1}\cdot\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$
$$2\frac{dz}{z}=\frac{d\lambda_1}{c^2+\lambda_1}\cdot\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$

since
$$2\frac{dy}{y}=\frac{d\lambda_1}{b^2+\lambda_1}+\frac{d\lambda_2}{b^2+\lambda_2}+\frac{d\lambda_3}{b^2+\lambda_3},\ \&\text{c.}$$

From these we see that the path of the particle lies always in a plane passing through the axis of x; by taking values of λ_1 so large that the fourth powers of the ratios $\sqrt{\frac{a^2}{\lambda_1}}$, $\sqrt{\frac{b^2}{\lambda_1}}$, $\sqrt{\frac{c^2}{\lambda_1}}$, can be neglected, Clebsch shewed that the path of particles very rapidly approached straight lines. By subtracting the second of these equations from the first and integrating we have

33. $$\log\frac{y}{z}=-(b^2-c^2)\int\frac{d\lambda_1}{(b^2+\lambda_1)(c^2+\lambda_1)\left\{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\right\}}.$$

This can be expressed in terms of the elliptic functions, for we have

$$\int\frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}=-\frac{1}{2}\frac{d\theta_1}{\sqrt{a^2-c^2}},$$
$$d\lambda_1=-\frac{1}{2}\frac{d\theta_1}{\sqrt{a^2-c^2}}\cdot\frac{\mathrm{sn}^3\theta_1}{\mathrm{cn}\theta_1\mathrm{dn}\theta_1},$$
$$(b^2+\lambda_1)(c^2+\lambda_1)=(a^2-c^2)^2\cdot\frac{\mathrm{cn}^2\theta_1\mathrm{dn}^2\theta_1}{\mathrm{sn}^4\theta_1},$$
$$A-2=2\left\{\frac{1}{k^2(a^2-c^2)^{\frac{3}{2}}}\left[\left(1-\frac{E}{K}\right)\theta_1-Z\theta_1\right]-1\right\}.$$

Making these substitutions, we have

34. $$\log\left(\frac{y}{z}\right)^{\frac{k'^2}{(a^2-c^2)^{\frac{1}{2}}}}=\int\frac{\mathrm{sn}^7\theta_1\,d\theta_1}{\mathrm{cn}^3\theta_1\,\mathrm{dn}^3\theta_1\left\{1-\left[\dfrac{\left(1-\dfrac{E}{K}\right)\theta_1-Z\theta_1}{k^2(a^2-c^2)^{\frac{3}{2}}}-1\right]\left[(a^2-c^2)^{\frac{3}{2}}\dfrac{\mathrm{cn}\,\theta_1\,\mathrm{dn}\,\theta_1}{\mathrm{sn}^3\theta_1}\right]\right\}}$$

a non-integrable expression. Nothing of interest could be obtained by a further examination of this case, and so we shall leave it, calling attention, however, to the fact that the coordinates of any fluid particle are expressed as functions of the parameter of the confocal ellipsoid upon which it lies. The same is also true in the case of a simple rotation around one axis—as the axis of x; the resulting integral equation of the path of the particle comes in this case also in a form that is too complicated to admit of any discussion. The motion of the particles, very near the surface of the body, would be, in an ordinary liquid, a very important point to determine—but, as in that case, the equations will present still greater difficulties than they do in the case of a perfect fluid, it does not seem as if that problem is susceptible of solution.

We shall now take up the second and more interesting part of the general problem and investigate the motion of the solid body. As there are no external forces acting upon either the body or the fluid, the motion of the system is entirely due to that of the body, and, in consequence, the total energy of the motion is a homogeneous quadratic function of the quantities u, v, w, p, q, r with constant coefficients. On account of the assumed symmetry of figure and distribution of mass of the body, *i. e.* with respect to three mutually perpendicular planes, the expression for the energy of the system will contain only the squares of u, v, &c., and may be written

$$2T = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2,$$

where the constant coefficients a_{ij} depend upon the density of the fluid, the form of the surface of the body, and the moments of inertia about the axes. Write for brevity

$$\frac{\partial T}{\partial u} = U, \ldots . \frac{\partial T}{\partial r} = R.$$

The Kirchhoffian equations for the motion of the body are now

35. $$\begin{aligned} \frac{\partial U}{\partial t} &= qW - rV, & \frac{\partial P}{\partial t} &= vW - wV + qR - rQ, \\ \frac{\partial V}{\partial t} &= rU - pW, & \frac{\partial Q}{\partial t} &= wU - uW + rP - pR, \\ \frac{\partial W}{\partial t} &= pV - qU, & \frac{\partial R}{\partial t} &= uV - vU + pQ - qP. \end{aligned}$$

From these equations we have the well known integrals

36. $$\begin{aligned} 2T &= \text{const.}, \\ U^2 + V^2 + W^2 &= \text{const.} = L, \\ UP + VQ + WR &= \text{const.} = M. \end{aligned}$$

For the assumed case the equations of motion become

$$\begin{aligned}
&a_{11}\frac{\partial u}{\partial t} = a_{33}wq - a_{22}vr\,, \quad && a_{44}\frac{\partial p}{\partial t} = (a_{33} - a_{22})\,vw + (a_{66} - a_{55})\,qr\,, \\
37.\qquad &a_{22}\frac{\partial v}{\partial t} = a_{11}ur - a_{33}wp\,, \quad && a_{55}\frac{\partial q}{\partial t} = (a_{11} - a_{33})\,wu + (a_{44} - a_{66})\,pr\,, \\
&a_{33}\frac{\partial w}{\partial t} = a_{22}vp - a_{11}uq\,, \quad && a_{66}\frac{\partial r}{\partial t} = (a_{22} - a_{11})\,uv + (a_{55} - a_{44})\,pq\,.
\end{aligned}$$

Before going on to the integration of these equations for special cases, it will be desirable to determine the values of the constants a_{ij}. These quantities are made up of two parts depending, as they do, upon the energies of the solid and the fluid; that part of a_{11} which depends upon the solid merely is m, or the mass of the solid; and the same is true for a_{22} and a_{33}: the parts of a_{44}, a_{55}, a_{66} depending upon the body are

$$\frac{m}{5}(b^2 + c^2)\,, \quad \frac{m}{5}(c^2 + a^2)\,, \quad \frac{m}{5}(a^2 + b^2)\,.$$

For the parts depending upon the energy of the fluid we have, denoting any one of them by a'_{ij},

$$a'_{ij} = \rho\int \varphi_j \frac{\partial \varphi_i}{\partial u}\,d\sigma$$

(see article above cited, or Kirchhoff's *Math. Phys.*); the integration is of course to be extended over the entire surface of the ellipsoid.

The values of these constants have been already obtained by several writers, and I will merely write them down—they are:

$$\begin{aligned}
&a'_{11} = m\frac{\rho}{\rho_0}\frac{A}{B + C}, \\
38.\qquad &a'_{22} = m\frac{\rho}{\rho_0}\frac{B}{C + A}, \qquad \rho_0 \text{ denoting the density of the solid.} \\
&a'_{33} = m\frac{\rho}{\rho_0}\frac{C}{A + B}.
\end{aligned}$$

$$\begin{aligned}
&a'_{44} = \frac{m}{5}(b^2 + c^2)\left\{1 + \frac{\rho}{\rho_0}\cdot\frac{B - C}{B - C + \dfrac{b^2 - c^2}{b^2 + c^2}(A + B + C)}\right\}, \\
39.\qquad &a'_{55} = \frac{m}{5}(c^2 + a^2)\left\{1 + \frac{\rho}{\rho_0}\cdot\frac{C - A}{C - A + \dfrac{c^2 - a^2}{c^2 + a^2}(A + B + C)}\right\}, \\
&a'_{66} = \frac{m}{5}(a^2 + b^2)\left\{1 + \frac{\rho}{\rho_0}\cdot\frac{A - B}{A - B + \dfrac{a^2 - b^2}{a^2 + b^2}(A + B + C)}\right\}.
\end{aligned}$$

The first three of these are the same in form as the values deduced by Mr. Ferrers in a recent number of the Quart. Jour.; the last three may be obtained from his values or from the values already given for N_1, N_2, N_3.

The values of these constants are thus seen to depend upon elliptic functions, and they can be readily determined from the previously given values of A_2, B_2, C_2, or they may be reduced to depend merely upon A, B, C. Consider for a moment the last three, viz: a_{44}, a_{55}, a_{66}; they are the moments of inertia of the ellipsoid, with respect to its axes, increased by certain quantities depending upon the density of the fluid and also upon the density and ellipsoidal shape of the immersed body. Conceive another ellipsoid whose semi-axes are a_1, b_1, c_1 and of mass m; the magnitude of the axes to be determined from the conditions

40.
$$\frac{m}{5}(b_1^2+c_1^2)=a_{44},$$
$$\frac{m}{5}(c_1^2+a_1^2)=a_{55},$$
$$\frac{m}{5}(a_1^2+b_1^2)=a_{66}.$$

Denote for brevity the fractional quantities in the expressions for a_{44}, a_{55}, a_{66} by ζ_1, ζ_2, ζ_3, then we have for the determination of a_1^2, b_1^2, c_1^2 the equations

41.
$$b_1^2+c_1^2=(b^2+c^2)(1+\zeta_1),$$
$$c_1^2+a_1^2=(c^2+a^2)(1+\zeta_2),$$
$$a_1^2+b_1^2=(a^2+b^2)(1+\zeta_3);$$

these give readily

42.
$$a_1^2=a^2+\frac{(a^2+b^2)\zeta_3+(c^2+a^2)\zeta_2-(b^2+c^2)\zeta_1}{2}\equiv a^2+\tau_1,$$
$$b_1^2=b^2+\frac{(a^2+b^2)\zeta_3-(c^2+a^2)\zeta_2+(b^2+c^2)\zeta_1}{2}\equiv b^2+\tau^2,$$
$$c_1^2=c^2+\frac{-(a^2+b^2)\zeta_3+(c^2+a^2)\zeta_2+(b^2+c^2)\zeta_1}{2}\equiv c^2+\tau_3;$$

and we have also for the new density

$$\rho'_0=\rho_0\frac{abc}{a_1b_1c_1}=\rho_0\frac{abc}{\sqrt{(a^2+\tau_1)(b^2+\tau_2)(c^2+\tau_3)}}.$$

If the ellipsoid is oblate

$$a_1^2+b_1^2=a^2+b^2,$$

if it is prolate,

$$b_1^2+c_1^2=b^2+c^2.$$

A further study of the properties of this ellipsoid would undoubtedly prove of interest, as it seems to possess, as will be seen shortly, properties somewhat analogous to "Poinsot's ellipsoid" in rigid dynamics.

The transformation to Lagrangian, or generalised coordinates, gives rise to several interesting results, and before going further with the discussion in Eulerian coordinates it will be desirable to make this transformation. I will write down at once the values of u, v, w, p, q, r in terms of the generalised coordinates $\dot{x}, \dot{y}, \dot{z}, \theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}$, as they are obtained in Thomson & Tait's *Nat. Phil.*, (new edition), and also in "*The Application of Generalised coordinates to the Kinetics of a Material System*," by Watson and Burbury; they are

$$\alpha. \quad \begin{aligned} u &= (\cos\theta\cos\phi\cos\psi - \sin\theta\sin\psi)\,\dot{x} \\ &\quad + (\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi)\,\dot{y} - \sin\theta\cos\phi\dot{z}\,, \\ v &= -(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\,\dot{x} \\ &\quad + (\cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi)\,\dot{y} + \sin\theta\sin\phi\dot{z}\,, \\ w &= \sin\theta\cos\psi\dot{x} + \sin\theta\sin\psi\dot{y} + \cos\theta\dot{z}\,, \end{aligned}$$

and for the rotations the well known values

$$\beta. \quad \begin{aligned} p &= \sin\phi\dot{\theta} - \sin\theta\cos\phi\dot{\psi}\,, \\ q &= \cos\phi\dot{\theta} + \sin\theta\sin\phi\dot{\psi}\,, \\ r &= \dot{\phi} + \cos\theta\dot{\psi}\,. \end{aligned}$$

The Lagrangian equations of motion are also, no external forces acting,

$$\gamma. \quad \begin{aligned} &\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial \dot{x}} = 0\,, \quad \frac{\partial}{\partial t} - \frac{\partial T}{\partial \dot{y}} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial \dot{z}} = 0\,, \\ &\frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial \varphi} - \frac{\partial T}{\partial \dot{\varphi}} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = 0\,; \end{aligned}$$

these equations can be found in either of the above mentioned books, or in Routh's Rigid Dynamics, and are too well known to need any explanation. The values of $\dot{x}, \dot{y}, \dot{z}$ could be obtained by direct solutions of equations α, but that would be a long process, and it is obvious from their forms that the desired quantities can be otherwise obtained. Multiply u by $\cos\theta\cos\phi$, v by $\cos\theta\sin\phi$, w by $\sin\theta$; and again multiply u by $\sin\phi$, v by $\cos\phi$; and finally u by $-\sin\theta\cos\phi$, v by $\sin\theta\cos\phi$, and w by $\cos\theta$ and in each case add the results; we have then

$$\begin{aligned} \dot{x}\cos\psi + \dot{y}\sin\psi &= u\cos\theta\cos\phi - v\cos\theta\sin\phi + w\sin\theta \\ -\dot{x}\sin\psi + \dot{y}\cos\psi &= u\sin\phi + v\cos\phi\,, \\ \dot{z} &= -u\sin\theta\cos\phi + v\sin\theta\cos\phi + w\cos\theta \\ &= (v-u)\sin\theta\cos\phi + w\cos\theta\,, \end{aligned}$$

the values of $\dot{x}$, $\dot{y}$ and $\dot{z}$ obtained from these equations are readily found to be

44.
$$\begin{aligned}\dot{x} &= (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\,u \\ &\quad - (\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\,v + \sin\theta\cos\psi w\,, \\ \dot{y} &= (\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi)\,u \\ &\quad - (\cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi)\,v + \sin\theta\sin\psi w\,, \\ \dot{z} &= -u\sin\theta\cos\phi + v\sin\theta\cos\phi + w\cos\theta\,.\end{aligned}$$

Having thus determined $\dot{x}$, $\dot{y}$, $\dot{z}$ as functions of θ, ϕ, ψ, if we can now, from equations β, determine these latter quantities as functions of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, we shall be in condition to integrate these differential equations and so determine the path described by any point of the body during its motion through the fluid. We know that whatever motion the body may have at any time that we can conceive the motion as due to a properly applied *impulse* at the beginning of the motion. It will simplify the work to assume the axis of the impulse as coinciding with one of the axes of reference, as the axis of z; then calling L, see equations 36, the total momentum of the impulse we will have from equations α—the components of momentum in the directions of x and y being zero,

45.
$$\frac{\partial T}{\partial \dot{x}} = 0\,,\quad \frac{\partial T}{\partial \dot{y}} = 0\,,\quad \frac{\partial T}{\partial \dot{z}} = L,\ \text{(a constant)}.$$

We have further, Thomson & Tait, § 221, for the Eulerian components of momentum $a_{11}u$, $a_{22}v$, $a_{33}w$

46.
$$\begin{aligned}a_{11}u\cos\phi - a_{22}v\sin\phi &= -L\sin\theta\,, \\ a_{11}u\sin\phi + a_{22}v\cos\phi &= 0\,, \\ a_{33}w &= L\cos\theta\,;\end{aligned}$$

from these follow

47.
$$\begin{aligned}a_{11}u &= -L\sin\theta\cos\phi\,, \\ a_{22}v &= L\sin\theta\sin\phi\,, \\ a_{33}w &= L\cos\theta\,.\end{aligned}$$

From these we have, for the first three terms in the expression for the energy,

48.
$$a_{11}u^2 + a_{22}v^2 + a_{33}w^2 = L^2\left[\sin^2\theta\left(\frac{\cos^2\varphi}{a_{11}} + \frac{\sin^2\varphi}{a_{22}}\right) + \frac{\cos^2\theta}{a_{33}}\right] \equiv G\,.$$

For the last three terms containing the squares of the angular velocities,

49. $a_{44}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 + a_{55}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 + a_{66}(\dot{\psi}\cos\theta + \dot{\phi})^2$;

this quantity is identical with the expression for the energy of a free rigid body rotating about its center of inertia (Thomson & Tait, p. 314), the

moments of inertia being a_{44}, a_{55}, a_{66}. We have for the total kinetic energy,

50. $$2T = G + a_{44}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 + a_{55}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 + a_{66}(\dot{\psi}\cos\theta + \dot{\phi})^2;$$

from this it is clear that the motion of the ellipsoid in the fluid is identical with the motion of another ellipsoid of semi-axes a_1, b_1, c_1 rotating about its center under a potential G, or, the entire kinetic energy of the system of fluid and ellipsoid is equal to the entire energy of an ellipsoid of semi-axes a_1, b_1, c_1 and density ρ'_0 rotating about its center under a potential G.

If the ellipsoid be of revolution around the axis of z the expressions for $\dot{x}$, $\dot{y}$, $\dot{z}$ become much simplified; we have in that case

$$a_{11} = a_{22}, \quad a_{44} = a_{55},$$

and, as we have already seen,

$$a_{66} = \frac{m}{5}(a^2 + b^2).$$

The values of $\dot{x}$, $\dot{y}$, $\dot{z}$ are now

51. $$\begin{aligned} \dot{x} &= -L\sin\theta\cos\theta\cos\psi\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right), \\ \dot{y} &= L\sin\theta = \cos\theta\sin\psi\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right), \\ \dot{z} &= \quad L\sin^2\theta\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right) + \frac{L}{a_{33}}. \end{aligned}$$

[A very interesting problem is solved by Thomson & Tait, viz: to find the motion of a solid of revolution in a fluid which so moves that the axis of revolution shall always be in one plane—the body has a motion of translation in the direction of the axis of x and an angular motion around this axis—and it is found that the resulting equation of motion is the same in form as that of the common pendulum, the mass, moment of inertia and length of the pendulum depending upon the mass and figure of the body and the density of the liquid.]

The substitution of the relations $a_{11} = a_{22}$ and $a_{44} = a_{55}$ in the expression for $2T$ cause this quantity to become

52. $$2T = L^2\sin^2\theta\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right) + \frac{L^2}{a_{33}} + a_{44}(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + a_{66}(\dot{\phi} + \cos\theta\dot{\psi})^2$$

containing neither ϕ nor ψ, and, consequently, cause the fifth and sixth of equations γ to become

$$\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\varphi}} = 0, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\psi}} = 0,$$

or,

$$\frac{\partial T}{\partial\dot{\varphi}} = \text{const.} \quad \frac{\partial T}{\partial\dot{\psi}} = \text{const.};$$

equating then to constants these derivatives of T and we have

53.
$$\dot{\phi} + \cos\theta\dot{\psi} = \mu_1,$$
$$a_{44}\sin^2\theta\dot{\psi} + a_{66}\mu_1\cos\theta = \mu_2,$$

μ_1 denoting the angular velocity around the axis of z and μ_2 the component angular momentum about the same axis. The equation

$$\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\theta}} = \frac{\partial T}{\partial\theta},$$

now merely expresses what we already know, that T, the kinetic energy, is constant.

The expression for the energy may now be written

$$L^2\sin^2\theta\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right) + \frac{L^2}{a_{33}} + a_{44}(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + a_{66}\mu_1^2.$$

The steady motion of the solid is easily obtained from the original expression for the energy, together with the values already given for u, v, w, p, q, r and for $\dot{x}$, $\dot{y}$ and $\dot{z}$. We must have $\theta = 0$ and consequently $\frac{\partial T}{\partial\theta} = 0$; since $\frac{\partial T}{\partial\varphi}$ and $\frac{\partial T}{\partial\psi}$ are identically zero, they of course do not come into the question, we must also have $\dot{\psi} = \text{const.}$, say η.

$$\frac{\partial T}{\partial\theta} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial\theta} + \ldots\ldots\frac{\partial T}{\partial r}\frac{\partial r}{\partial\theta};$$

$$\frac{\partial T}{\partial u} = a_{11}u, \quad \frac{\partial T}{\partial p} = a_{44}p,$$
$$\frac{\partial T}{\partial v} = a_{33}v, \quad \frac{\partial T}{\partial q} = a_{55}q,$$
$$\frac{\partial T}{\partial w} = a_{33}w, \quad \frac{\partial T}{\partial r} = a_{66}r;$$

since $a_{22} = a_{33}$ and $a_{55} = a_{66}$.

$$\frac{\partial u}{\partial\theta} = -(\sin\theta\cos\phi\cos\psi\dot{x} + \sin\theta\cos\phi\sin\psi\dot{y} + \cos\theta\cos\phi\dot{z}),$$
$$\frac{\partial v}{\partial\theta} = \quad(\sin\theta\sin\phi\cos\dot{\psi} + \sin\theta\sin\phi\sin\psi\dot{y} + \cos\theta\sin\phi\dot{z}),$$
$$\frac{\partial w}{\partial\theta} = \quad(\cos\theta\cos\psi\dot{x} + \cos\theta\sin\psi\dot{y} = \sin\theta\dot{z}),$$
$$\frac{\partial p}{\partial\theta} = -\cos\theta\cos\phi\dot{\psi},$$
$$\frac{\partial q}{\partial\theta} = \quad\cos\theta\sin\phi\dot{\psi},$$
$$\frac{\partial r}{\partial\theta} = -\sin\theta\dot{\psi},$$

Substituting all these values in $\frac{\partial T}{\partial \theta}$, and also giving p and q their values in the generalized coordinates, and this becomes

$$\begin{aligned}\frac{\partial T}{\partial \theta} = & -a_{11}(u\cos\phi - v\sin\phi)(\sin\theta\cos\psi\dot{x} + \sin\psi\dot{y} + \cos\theta\dot{z}) \\ & + a_{33}w(u\cos\phi - v\sin\phi) \\ & - a_{44}(\sin\phi\dot{\theta} - \sin\theta\cos\phi\dot{\psi})\cos\theta\cos\phi\dot{\psi} \\ & + a_{44}(\cos\phi\dot{\theta}\text{ m }\sin\theta\sin\phi\dot{\psi})\cos\theta\sin\phi\dot{\psi} \\ & + a_{66}r\sin\theta\dot{\psi},\end{aligned}$$

since

$$\begin{aligned}u\cos\phi - v\sin\phi = & \dot{x}\cos\psi\cos\theta + \dot{y}\sin\psi\cos\theta - \dot{z}\sin\theta \\ = & (a_{33} - a_{11})(u\cos\phi - v\sin\phi)w \\ & + a_{44}\sin^2\theta\dot{\psi}^2 - a_{66}r\cos\theta\dot{\psi} \\ = & L^2\sin\theta\cos\theta\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) + a_{44}\sin\theta\cos\theta\dot{\psi}^2 - a_{66}r\sin\theta\dot{\psi}.\end{aligned}$$

Calling now the integral of $\theta' = 0$, $\theta = \theta'$, then $\cos\theta = \cos\theta'$, we had $\dot{\psi} = \eta$, therefore, $\cos\psi = \cos\eta t$, &c. The equation $\frac{\partial T}{\partial \theta} = 0$, becomes now

54. $$a_{44}\cos\theta'\eta^2 - a_{66}r\eta + L^2\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)\cos\theta' = 0.$$

which gives

55. $$r = \frac{L^2\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)\cos\theta'}{a_{66}\eta} + \frac{a_{44}}{a_{66}}\cos\theta\eta,$$

the value of the angular velocity necessary to the maintaining of steady motion when the body is projected in the direction of its axis. The equations for $\dot{x}$, $\dot{y}$, $\dot{z}$ now give after integration

56. $$\begin{aligned}x &= \frac{L}{\eta}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)\sin\theta'\cos\theta'\sin\eta t, \\ y &= \frac{L}{\eta}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)\sin\theta'\cos\theta'\cos\eta t, \\ z &= L\left(\frac{\cos^2\theta'}{a_{33}} + \frac{\sin^2\theta'}{a_{11}}\right)t,\end{aligned}$$

or

57. $$\begin{aligned}\frac{x}{y} &= \tan\eta t \\ z &= L\cos^2\theta'\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)t + \frac{Lt}{a_{11}},\end{aligned}$$

the equations of a helix whose projection on the plane of xy is the circle

58. $$x^2 + y^2 = \frac{L^2}{\eta^2}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)^2 \sin^2\theta' \cos^2\theta'.$$

This is in accordance with what was proved in the paper already referred to, viz: that a body moving in a fluid has an infinite number of possible steady motions, each of which consists of a twist about a certain screw.

The problem of the motion of a solid of revolution in a fluid has been very fully solved by Kirchhoff in Borchardt's Journal, Bd. 71, and by others since that time. I append a list of the more important articles on this subject:

Clebsch: Die Bewegung eines Ellipsoids in einer Flüssigkeit. Crelle's Journal, Bd. 52 and 53.

Kirchhoff: Ueber die Bewegung eines Rotations Körpers in einer Flüssigkeit. Borchardt's Journal, Bd. 71.

Ferrers: The motion of an infinite mass of water about a moving Ellipsoid. Quart. Jour., No. 52. 1875.

Köpcke: Zur Discussion der Bewegung eines Rotations Körpers in einer Flüssigkeit. Math. Annalen., Vol. 12, p. 387.

Weber: Anwendung der Thetafunctionen zweier Veränderlichen auf die Theorie der Bewegung eines festen Körpers in einer Flüssigkeit. Math. Annalen., Vol. 14, p. 173.

The motion of a sphere in a fluid was treated by Dirichlet, and was the first article of note which treated of the motion in a fluid of a body of given form.

WASHINGTON, *July 5, 1879.*

On Certain Ternary Cubic-Form Equations.

By J. J. Sylvester.

Chapter I. *On the Resolution of Numbers into the sums or differences of Two Cubes.*

Section 1.

M. Lucas has written to inform me that in some one or more of a series of memoirs commencing with 1870, or elsewhere, the Reverend Father Pépin has made considerable additions to my published theorems on the classes of numbers irresoluble into the *sum or difference** of two rational cubes.

Using p, q to denote primes of the forms $18n+5$, $18n+11$, besides the 6 forms published by me, M. Pépin has found 10 other general classes of irresoluble numbers, the total number (as I understand from M. Lucas) known to the Reverend Father being as follows:

$$\begin{array}{cccccccc} p, & q^2, & p^2, & q, & 2p, & 2q^2, & 4p^2, & 4q, \\ 9p, & 9q^2, & 9p^2, & 9q, & 25p, & 25q^2, & 5p^2, & 5q, \end{array}$$

but the last four of these classes are special cases only, of three out of the four more general irresoluble classes pq, p^2q^2, $p_1p_2^2$, $q_1q_2^2$, where p_1, p_2 are any two numbers of the p class and q_1, q_2 any two of the q class. On making $p=5$ in the first two of these, and $p_1=5$, $p_2=p$, or $p_2=5$, $p_1=p$, in the third, Father Pépin's last four classes result. It is also true that the numbers in my four additional general classes respectively multiplied by 9 are still irresoluble. Hence the number of known classes of numbers (depending on p and q) irresoluble into the sum or difference of cubes may be arranged as follows:

$$\begin{array}{cccccccc} p, & q, & p^2, & q^2, & pq, & p^2q^2, & p_1p_2^2, & q_1q_2^2, \\ 9p, & 9q, & 9p^2, & 9q^2, & 9pq, & 9p^2q^2, & 9p_1p_2^2, & 9q_1q_2^2, \\ 2p, & 4q, & 4p^2, & 2q^2. & & & & \end{array}$$

Moreover, I have ascertained the truth of the following two theorems of a somewhat different character:

1°. Let ρ, ψ, ϕ denote prime numbers respectively of the forms $18n+1$, $18n+7$, $18n+13$ and suppose ρ, ψ, ϕ to be *not* of the form f^2+27g^2 and

*It is well to understand that a number resoluble into the sum is necessarily also resoluble into the difference of two positive cubes and *vice versâ*.

consequently *not* to possess the cubic residue 2, then I say that all the numbers comprised in any one of the eight classes

$$2\rho,\ 4\rho,\ 2\rho^2,\ 4\rho^2,\ 2\psi,\ 4\psi^2,\ 4\phi,\ 2\phi^2$$

are irresoluble into the sum of two cubes.*

2°. Provided 3 is not a cubic residue to ν† [where ν, any $6n+1$ prime, is the same as ρ, ϕ, ψ taken collectively], 3ν and $3\nu^2$ are similarly irresoluble.

With the aid of these theorems and certain special cases of irresolubility noticed by Father Pépin, communicated to me by M. Lucas, supplemented by calculations of M. Lucas and my own as regards the non-excluded numbers, it follows (*mirabile dictu*) that of the first 100 of the natural order of numbers, there is only a single one, viz. 66, of which it cannot at present be affirmed with certitude either that it is or is not resoluble into the sum of two cubes, and of which, in the former case, the resolution cannot be exhibited.

The proof of these statements, and the resolutions into cubes in their lowest terms, when they exist, will be given in the next number of the Journal. For the present I limit myself to noticing (what I much regret not to have done before the paper was printed) a statement of M. Lucas which is capable of being misunderstood and might give rise to an erroneous conception.

It is where this distinguished contributor to our Journal speaks of deriving from one rational point on a cubic curve (defined by a cubic equation with integer coefficients) another by means of its intersections with a conic drawn through five consecutive points situated at the given rational one; but, in fact, it follows from my theory of *residuation* that this point is

* The exclusion of 2 as a cubic residue blocks out the possibility of the "distribution of the amplitude;" the form p^2+27q^2 blocks out the possibility of a solution in which x^2-xy+y^2 has a common factor with the amplitude, and thereby imposes upon the equation containing x, y, z (were it soluble in integers) the necessity of repeating itself perpetually in with smaller numbers, which of course is impossible. But the two conditions thus separately stated are in fact mutually implicative, every number of the form f^2+27g^2 having 2 for a cubic residue and *vice versâ* every number of the form $6n+1$ to which 2 is a cubic residue being of the form f^2+27g^2. The sole condition, therefore, in order that a number coming under any of the eight categories in the text shall be known at sight to be irresoluble into the sum of two cubes, is that its variable part shall not be of the form p^2+27q^2, *i. e.* shall not be 31, 43, 91, 109, 127, 157, 223, 229, 247, etc.

† If I am not mistaken this is tantamount to the proviso that ν shall not be of the form $f^2\pm 9fg+81g^2$. It is worth noticing that the above quantity multiplied by 3, say $3N$, is equal to $\dfrac{(9g\mp f)^3+(18g\pm f)^3}{27g}$, so that when g is a cube number N is immediately resoluble. The initial values of N will be found to be 61, 67, 73, 103, 151, 193, 271, 367, 547, etc., for each of which, up to 367 inclusive, $g=1$ or $g=-1$, so that their products by 3 are immediately resoluble.

collinear with the given point and its second tangential: just as a ninth point in which the cubic would be met by any other cubic passing through *eight* consecutive points situated at the given point would be the third tangential to the latter.*

Hence M. Lucas' third method amounts only to a combination of the other two; and in fact there is *but one single scale* of rational derivatives from any given point in a general cubic, the successive terms of which expressed in terms of the coordinates of the primitive are of the orders 1, 4, 16, 25, 49, . . . the squares of the natural numbers with the multiples of 3 omitted.†

Scholium.

I term lmn the *amplitude* of the equation $lx^3 + my^3 + nz^3 = 0$, and if A cannot be broken up in any way into factors l, m, n, such that $lx^3 + my^3 + nz^3 = 0$ shall be soluble in integers, I call the amplitude A of the equation $x^3 + y^3 + Az^3 = 0$ *undistributable.*

When A is of the form $\dfrac{x^3 - 3x^2y + y^3}{3z^3}$, the equation $x^3 + y^3 + Az^3 = 0$ is always soluble, and when this equation is soluble, then, provided that its amplitude is undistributable and contains no prime factor of the form $6i + 1$, the equation $x^3 - 3x^2y + y^3 = 3Az^3$ must be soluble in integers, which cannot be the case when A contains any factor other than 3, or of the form $18i \pm 1$, inasmuch as *the cubic form* $x^3 - 3x \pm 1$ *contains no factors other than* 3 *or of the form* $18i \pm 1$.

This last theorem is a particular case of the following: If k be any integer and $F(x\, y)$, the product of factors of the form $\left(x - 2\cos\frac{\lambda\pi}{k}\, y\right)$, where

*I make the important additional remark that at those special points of the cubic where this ninth point (sometimes elegantly called the subosculatrix) coincides with the point osculated, the scheme of rational derivatives returns upon itself, and instead of an infinite number there will be only two rational derivatives to such point. That is to say the infinite scheme becomes a system of 3 continually recurring points. The general theory of the special points which have only a finite number of rational derivatives will be given in the next number of the Journal.

†When the cubic is of the form $Ax^3 + Ay^3 + Cz^3 + Mxyz = 0$, where A, C, M are integers, then a rational point of inflection $x = 1, y = -1, z = 0$ is known and, in that case, from any other rational point *besides the ordinary ones* derivative rational points of the missing orders 9, 36, 81 can be found, but no others, and so universally if in the general cubic a rational point of inflection and a rational point (a, b, c) are given the scale of rational derivatives will be of the orders 1, 4, 9, 16, . . . in a, b, c. This scale will of course be duplex, consisting of a series of points and a second series in which the radii drawn through the points of the first series and the point of inflection again meet the cubic.

λ is every number prime to k up to $\frac{k-1}{2}$, then Fx contains no prime factors excepting such as are contained in k or else are of the form $ki \pm 1$.*

If it could be shown, in analogy with what holds for the quadratic forms Fx which result from making $k = 8$, 10, 12, that the cubic form $x^3 - 3xy^2 \pm y^3$ which results from making $k = 18$ may always be made to represent any prime number of the form $18n \pm 1$ itself, or else its treble (and for our purpose rational numbers would be as efficient as integers), we should then be able to affirm that any prime $18n \pm 1$ or else its nonuple could be resolved into the sum of two cubes. As a matter of fact I have ascertained that every prime number $18n \pm 1$ as far as 537 inclusive (and have no ground for supposing that the law fails at that point) can be represented by $x^3 - 3xy^2 \pm y^3$ or else by its third part with *integer* values of x, y. Moreover, I find that the same thing is true of 17^2, $17 . 19$, 19^2, $17 . 37$, $19 . 37$, 37^2, $17 . 53$, $19 . 53$, $37 . 53$, *i. e.* in fact for all the binary combinations of the natural progression of "r, ρ" numbers 17, 19, 37, 53, 71, 73, 89 (21 in all), as also 17^2, 19^2, 37^2.† The number of *consecutive* r, ρ primes for which the law has been verified, *i. e.* the number of those not exceeding 537 will be found to be 39, viz: 17, 19, 37, 53, 71, 73, 89, 107, 109, 127, 163, 179, 181, 197, 199, 233, 251, 269, 271, 307, 323, 341, 359, 361, 377, 379, 397, 413, 431, 433, 449, 451, 467, 469, 487, 503, 521, 523, 541, which according to the usual canons of induction would, I presume, be considered almost sufficient to establish the theorem for the case of $k = 9$.

The table of "*special* cases" of irresoluble numbers found by Father Pépin (according to the information most kindly communicated to me by

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* Thus, by making $k = 8$ we learn that $x^2 - 2$ contains no factors except 2 and $8i \pm 1$ and by making $k = 16$, that $y^4 - 4y^2 + 2$, none except 2 or $16i \pm 1$, by making $k = 9$ that $x^3 - 3x + 1$, by making $k = 18$, that $x^3 - 3x - 1$ contain no other factors but 3, or numbers of the form $18n \pm 1$. The theorem, I am aware, is well known for the case where k is a prime number and possibly is so for the general case. The proof of the irresolubility into two cubes of the 20 classes of numbers involving p's and q's, given at page 280, is an instantaneous consequence of the theorem for the case of $k = 9$, for which case also there is no shadow of doubt of the theorem being true.

† 53^2 has not yet made its appearance. All the primes of that form themselves occurring in the first six hundred numbers have already occurred in my calculations except 557 and 593. I have worked with the formula $x^3 - 3xy^2 \pm y^3$ [x and y relative primes], giving to x and to y all the values possible from 1 to 36, and intend to extend the table to the limit of 50 or 100. The longer a moderate-sized number is in making its appearance, the longer it is likely to be before it appears, inasmuch as the large numbers of which it is the residuum or balance are becoming continually greater. It may very well then happen that the missing numbers alluded to may transcend all practicable limits of calculation to find them just as would be the case, for certain values of A, with finding values of x, y to satisfy the Pellian equation $x^2 - Ay^2 = 1$, were there not a theoretical method of arriving at them.

M. Lucas) comprises the numbers

14, 21, 31, 38, 39, 52, 57, 60, 67, 76, 77, 93, 95,*

all of which I have verified as irresoluble except the number 60, which I accept as such on the erudite and sagacious Father's authority.

Reverting to F, it is hardly necessary to recall that $F(x^2+y^2, xy)$ is the primitive factor of x^k-y^k, and that it is capable of very easy demonstration that this primitive factor contains no prime factors except such as are divisors of k or of the form $ki+1$, the linear divisor $ki-1$ being here excluded. It seems to be very probable that for $k=9$, $F(x, y)$ or else $3F(x, y)$ does represent any prime of the form $18n \pm 1$, and consequently that every such form of prime or else 9 times the same is the sum of two rational cubes.†

This last conjectural theorem, it will be noticed, is not in any real analogy to the theorem that every product of primes of the form $4n+1$, and also the double thereof, is the sum of two *integer* squares; the real analogy is between the fact, of which this theorem is a consequence, that $x^3-3xy^2 \pm y^3$ or its third part represents every number which is a product of primes of the form $18n \pm 1$, and each one of the facts that x^2-2y^2, x^2-5y^2 represent all numbers of the form $8i \pm 1$, $10i \pm 1$ respectively, and that x^2-3y^2 or its third part represents all numbers of the form $12i \pm 1$. On account of its importance to this theory it seems desirable to give a name to the law which governs the prime factors of $F(x, y)$, and I take advantage of the circumstance that $F(x^2+y^2, xy)$ contains prime factors of the form $ki+1$, but not of the form $ki-1$, whilst $F(x, y)$ contains prime factors of either of these forms indifferently, to characterize it as the Law of Twin Prime Factors. Let us suppose the circumference of a circle divided by points into k equal parts, and agree to designate the shorter arc between any two of the points a *primitive* division of the circle in respect to k, provided that no number less than k would be adequate to give rise to an equal length of arc, so that $\frac{2\lambda\pi}{k}$, when λ is prime to k and less than $\frac{k}{2}$, will serve to represent any such division. The assumed Law of Twin Factors (well known, I repeat, for the case of k a prime number and possibly in its extended form likewise) may then be enunciated as follows:

* Of these numbers all except 60, 31, 67, 77, 95 belong to some one or other of the general classes of irresoluble numbers given in the text.

† It may be and probably is true also that $x^3-3xy^2 \pm y^3$ will represent the product or else three times the product of any two primes each of which is of the form r or ρ, and possibly the square or else three times the square of any r or ρ; it cannot possibly represent three times *any cube*, for if it did we should be able to infer that a cube was resoluble into two cubes, which we know is not true.

That function of x whose first coefficient is unity and whose roots are the doubled cosines of all the primitive divisions of the circle in respect to k contains no prime factors except such as are divisors of, or else when increased or diminished by unity, are divisible by k. This may be called again the *Exclusional or Negative Theorem of Twin Factors;* and on the other hand the more extraordinary theorem which asserts (on evidence not yet conclusive) that the function of x above defined, when made homogeneous in x, y, will represent (at all events for the case of $k=9$) every prime number of the form $ki \pm 1$, or else certain specific multiples of any such number, may be called the *Inclusional or Representational Theorem of Twin Factors.*

A New Proof of the Theorem of Reciprocity.

BY DR. JULIUS PETERSEN, *of Copenhagen, Denmark.*

LET a and b be two odd prime numbers, $a < b$; form the equation

$$(2n+1)\,a - 2mb = r \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

for all the odd numbers $2n+1$ up to $p-2$, and choose m so that r, which will be termed a remainder, shall lie between $-b$ and $+b$. The absolute numerical values of these remainders will be all different, and therefore they must be the odd numbers $1, 3, \ldots, (b-2)$, of which, however, some may be negative. According as the number of these negative values is even or odd we will write

$$\left(\frac{a}{b}\right) = +1 \quad \text{or} \quad \left(\frac{a}{b}\right) = -1.$$

From among the equations (1) we will take out those in which the remainders lie between $-a$ and $+a$; there is one such equation for every value of $2m$ up to $a-1$; these equations may be written under the form

$$(a-2m)\,b - (b-2n-1)\,a = r, \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

which are evidently the equations for the determination of the sign of $\left(\frac{b}{a}\right)$. Therefore $\left(\frac{a}{b}\right)$ and $\left(\frac{b}{a}\right)$ will have the same or opposite signs, according as the number of the remainders between $-b$ and $-a$ is even or odd. For such remainders

$$-b < (2n+1)\,a - 2mb < -a,$$

or, putting $m = n - k$, $a = b - 2\alpha$,

$$2n + 2 > \frac{(k+1)b}{a} > 2n + 1 .$$

Therefore there is a negative remainder between $-a$ and $-b$ for every term in the series

$$\frac{b}{a}, \quad \frac{2b}{a}, \quad \frac{3b}{a}, \ldots, \frac{(a-1)b}{a}$$

in which the greatest contained integer is odd; now any two terms in the above series at equal distances from its extremities have the sum b, and therefore the integer parts of such two (their sum being $b-1$) are both even or both odd; for a odd the number of terms is even, and the number of negative remainders between $-a$ and $-b$ therefore even; for a even the number depends upon the mean term $\frac{1}{2}b$; it is even if $b = 4i + 1$, odd if $b = 4i + 3$; hence we shall always have

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2} \cdot \frac{b-1}{2}} .$$

By multiplying the equations (1) by c we obtain

$$\left(\frac{ac}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{c}{b}\right);$$

and, remarking that, b being a prime number, the multiplication* of the equations (1) shows that $\left(\frac{a}{b}\right)$ is in accord with Legendre's notation, we have the proof of the extended (erweitert) theorem.

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* Viz: multiplying together the equations (1) in the form

$$\begin{aligned} 1 \,.\, a &\equiv r_1 \quad (\text{mod. } 2b), \\ 3 \,.\, a &\equiv r_2 \quad (\;\;\text{"}\;\;), \\ 5 \,.\, a &\equiv r_3 \quad (\;\;\text{"}\;\;), \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ (b-2) \,.\, a &\equiv r_{\frac{b-1}{2}} \quad (\;\;\text{"}\;\;), \end{aligned}$$

and noticing that $r_1 \,.\, r_2 \,.\, r_3 \ldots r_{\frac{b-1}{2}} = \left(\frac{a}{b}\right) 1 \,.\, 3 \,.\, 5 \ldots (b-2)$, we have

$$a^{\frac{b-1}{2}} \equiv \left(\frac{a}{b}\right) (\text{mod. } 2b),$$

and thus $\left(\frac{a}{b}\right) = +1$ or -1 is the criterion that a shall be a quadratic residue or non-residue of b.

EDS.

On a New Action of the Magnet on Electric Currents.

By E. H. Hall, *Fellow of the Johns Hopkins University.*

Sometime during the last University year, while I was reading Maxwell's Electricity and Magnetism in connection with Professor Rowland's lectures, my attention was particularly attracted by the following passage in Vol. II, p. 144:

"It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. If the conductor be a rotating disk or a fluid it will move in obedience to this force, and this motion may or may not be accompanied with a change of position of the electric current which it carries. But if the current itself be free to choose any path through a fixed solid conductor or a network of wires, then, when a constant magnetic force is made to act on the system, the path of the current through the conductors is not permanently altered, but after certain transient phenomena, called induction currents, have subsided, the distribution of the current will be found to be the same as if no magnetic force were in action. The only force which acts on electric currents is electromotive force, which must be distinguished from the mechanical force which is the subject of this chapter."

This statement seemed to me to be contrary to the most natural supposition in the case considered, taking into account the fact that a wire not bearing a current is in general not affected by a magnet and that a wire bearing a current is affected exactly in proportion to the strength of the current, while the size and, in general, the material of the wire are matters of indifference. Moreover in explaining the phenomena of statical electricity it is customary to say that charged bodies are attracted toward each other or the contrary solely by the attraction or repulsion of the charges for each other.

Soon after reading the above statement in Maxwell I read an article by Prof. Edlund, entitled "*Unipolar Induction*" (Phil. Mag., Oct., 1878, or Annales de Chemie et de Physique, Jan., 1879), in which the author evi-

dently assumes that a magnet acts upon a current in a fixed conductor just as it acts upon the conductor itself when free to move.

Finding these two authorities at variance, I brought the question to Prof. Rowland. He told me he doubted the truth of Maxwell's statement and had sometime before made a hasty experiment for the purpose of detecting, if possible, some action of the magnet on the current itself, though without success. Being very busy with other matters however, he had no immediate intention of carrying the investigation further.

I now began to give the matter more attention and hit upon a method that seemed to promise a solution of the problem. I laid my plan before Prof. Rowland and asked whether he had any objection to my making the experiment. He approved of my method in the main, though suggesting some very important changes in the proposed form and arrangement of the apparatus. The experiment proposed was suggested by the following reflection:

If the current of electricity in a fixed conductor is itself attracted by a magnet, the current should be drawn to one side of the wire, and therefore the resistance experienced should be increased.

To test this theory, a flat spiral of German silver wire was inclosed between two thin disks of hard rubber and the whole placed between the poles of an electromagnet in such a position that the lines of magnetic force would pass through the spiral at right angles to the current of electricity.

The wire of the spiral was about ½ mm. in diameter, and the resistance of the spiral was about two ohms.

The magnet was worked by a battery of twenty Bunsen cells joined four in series and five abreast. The strength of the magnetic field in which the coil was placed was probably fifteen or twenty thousand times H, the horizontal intensity of the earth's magnetism.

Making the spiral one arm of a Wheatstone's bridge and using a low resistance Thomson galvanometer, so delicately adjusted as to betray a change of about one part in a million in the resistance of the spiral, I made, from October 7th to October 11th inclusive, thirteen series of observations, each of forty readings. A reading would first be made with the magnet active in a certain direction, then a reading with the magnet inactive, then one with the magnet active in the direction opposite to the first, then with the magnet inactive, and so on till the series of forty readings was completed.

Some of the series seemed to show a slight increase of resistance due to the action of the magnet, some a slight decrease, the greatest change indicated by any complete series being a decrease of about one part in a hundred and fifty thousand. Nearly all the other series indicated a very much smaller change, the average change shown by the thirteen series being a decrease of about one part in five millions.

Apparently, then, the magnet's action caused no change in the resistance of the coil.

But though conclusive, apparently, in respect to any change of resistance, the above experiments are not sufficient to prove that a magnet cannot affect an electric current. If electricity is assumed to be an incompressible fluid, as some suspect it to be, we may conceive that the current of electricity flowing in a wire cannot be forced into one side of the wire or made to flow in any but a symmetrical manner. The magnet may *tend* to deflect the current without being able to do so. It is evident, however, that in this case there would exist a state of stress in the conductor, the electricity pressing, as it were, toward one side of the wire. Reasoning thus, I thought it necessary, in order to make a thorough investigation of the matter, to test for a difference of potential between points on opposite sides of the conductor.

This could be done by repeating the experiment formerly made by Prof. Rowland, and which was the following:

A disk or strip of metal, forming part of an electric circuit, was placed between the poles of an electro-magnet, the disk cutting across the lines of force. The two poles of a sensitive galvanometer were then placed in connection with different parts of the disk, through which an electric current was passing, until two nearly equipotential points were found. The magnet current was then turned on and the galvanometer was observed, in order to detect any indication of a change in the relative potential of the two poles.

Owing probably to the fact that the metal disk used had considerable thickness, the experiment at that time failed to give any positive result. Prof. Rowland now advised me, in repeating this experiment, to use gold leaf mounted on a plate of glass as my metal strip. I did so, and, experimenting as indicated above, succeeded on the 28th of October in obtaining, as the effect of the magnet's action, a decided deflection of the galvanometer needle.

This deflection was much too large to be attributed to the direct action of the magnet on the galvanometer needle, or to any similar cause. It was,

moreover, a permanent deflection, and therefore not to be accounted for by induction.

The effect was reversed when the magnet was reversed. It was not reversed by transferring the poles of the galvanometer from one end of the strip to the other. In short, the phenomena observed were just such as we should expect to see if the electric current were pressed, but not moved, toward one side of the conductor.

In regard to the direction of this pressure or tendency as dependent on the direction of the current in the gold leaf and the direction of the lines of magnetic force, the following statement may be made:

If we regard an electric current as a single stream flowing from the positive to the negative pole, *i. e.* from the carbon pole of the battery through the circuit to the zinc pole, in this case the phenomena observed indicate that two *currents*, parallel and in the same direction, tend to repel each other.

If, on the other hand, we regard the electric current as a stream flowing from the negtive to the positive pole, in this case the phenomena observed indicate that two *currents* parallel and in the same direction tend to attract each other.

It is of course perfectly well known that two *conductors*, bearing currents parallel and in the same direction, are drawn toward each other. Whether this fact, taken in connection with what has been said above, has any bearing upon the question of the absolute direction of the electric current, it is perhaps too early to decide.

In order to make some rough quantitative experiments, a new plate was prepared consisting of a strip of gold leaf about 2 cm. wide and 9 cm. long mounted on plate glass. Good contact was insured by pressing firmly down on each end of the strip of gold leaf a thick piece of brass polished on the under side. To these pieces of brass the wires from a single Bunsen cell were soldered. The portion of the gold leaf strip not covered by the pieces of brass was about 5½ cm. in length and had a resistance of about 2 ohms. The poles of a high resistance Thomson galvanometer were placed in connection with points opposite each other on the edges of the strip of gold leaf and midway between the pieces of brass. The glass plate bearing the gold leaf was fastened, as the first one had been, by a soft cement to the flat end of one pole of the magnet, the other pole of the magnet being brought to within about 6 mm. of the strip of gold leaf.

The apparatus being arranged as above described, on the 12th of November a series of observations was made for the purpose of determining the variations of the observed effect with known variations of the magnetic force and the strength of current through the gold leaf.

The experiments were hastily and roughly made, but are sufficiently accurate, it is thought, to determine the law of variation above mentioned as well as the order of magnitude of the current through the Thomson galvanometer compared with the current through the gold leaf and the intensity of the magnetic field.

The results obtained are as follows:

Current through Gold Leaf Strip. C.	Strength of Magnetic Field. M.	Current through Thomson Galvanometer. c.	$\frac{C \times M}{c}$
.0616	11420 H	.00000000232	303000000000.
.0249	11240 "	085	329.........
.0389	11060 "	135	319.........
.0598	7670 "	147	312.........
.0595	5700 "	104	326.........

H is the horizontal intensity of the earth's magnetism $= .19$ approximately.

Though the greatest difference in the last column above amounts to about 8 per cent. of the mean quotient, yet it seems safe to conclude that with a given form and arrangement of apparatus the action on the Thomson galvanometer is proportional to the product of the magnetic force by the current through the gold leaf. This is not the same as saying that the effect on the Thomson galvanometer is under all circumstances proportional to the current which is passing between the poles of the magnet. If a strip of copper of the same length and breadth as the gold leaf but ½ mm. in thickness is substituted for the latter, the galvanometer fails to detect any current arising from the action of the magnet, except an induction current at the moment of making or breaking the magnet circuit.

It has been stated above that in the experiments thus far tried the current apparently tends to move, without actually moving, toward the side of the conductor. I have in mind a form of apparatus which will, I think, allow the current to follow this tendency and move across the lines of magnetic force. If this experiment succeeds, one or two others immediately suggest themselves.

To make a more complete and accurate study of the phenomenon described in the preceding pages, availing myself of the advice and assistance of Prof. Rowland, will probably occupy me for some months to come.

BALTIMORE, *Nov. 19th, 1879.*

It is perhaps allowable to speak of the action of the magnet as setting up in the strip of gold leaf a new electromotive force at right angles to the primary electromotive force.

This new electromotive force cannot, under ordinary conditions, manifest itself, the circuit in which it might work being incomplete. When the circuit is completed by means of the Thomson galvanometer, a current flows.

The actual current through this galvanometer depends of course upon the resistance of the galvanometer and its connections, as well as upon the distance between the two points of the gold leaf at which the ends of the wires from the galvanometer are applied. We cannot therefore take the ratio of C and c above as the ratio of the primary and the transverse electromotive forces just mentioned.

If we represent by E' the difference of potential of two points a centimeter apart on the transverse diameter of the strip of gold leaf, and by E the the difference of potential of two points a centimeter apart on the longitudinal diameter of the same, a rough and hasty calculation for the experiments already made shows the ratio $\frac{E}{E'}$ to have varied from about 3000 to about 6500.

The transverse electromotive force E' seems to be, under ordinary circumstances, proportional to Mv, where M is the intensity of the magnetic field and v is the *velocity* of the electricity in the gold leaf. Writing for v the equivalent expression $\frac{C}{s}$ where C is the primary current through a strip of the gold leaf 1 cm. wide, and s is the area of section of the same, we have $E' \propto \frac{MC}{s}$.

November 22d, 1879.

Tables of the Generating Functions and Groundforms for Simultaneous Binary Quantics of the First Four Orders, taken two and two together.

BY J. J. SYLVESTER, *assisted by* F. FRANKLIN,
Of the Johns Hopkins University.

IN the Generating Functions given below, the exponents of the letters a, b, c, d, refer to degree in the coefficients of the quantics of the 1st, 2nd, 3rd and 4th orders respectively; the exponents of the letter x to order in the variables. Where the system consists of two quantics of the same order, the Latin letter and the corresponding Greek letter have been used. In the tabulated numerators, the *minus* sign has been placed *over* the number which it affects.

In each of the systems considered in this paper, with the exception of that consisting of a cubic and a quartic, it is found that there is never more than one groundform of any given type (*i. e.* of a given order in the variables and given degrees in the coefficients of the quantics); where, therefore, in the enumeration of the groundforms, the *type* alone is given, the *number* of groundforms of the type is to be understood to be 1. The symbol (λ, μ) is used to indicate a form of the degrees λ and μ in the coefficients of the two quantics, the number placed first always relating to the quantic of lower order, when the orders are different. In the last three cases, the numbers, as well as the types, of the groundforms are given in tables, which require no explanation.

SYSTEM OF TWO LINEARS.*

G. F. for differentiants, $\dfrac{1}{(1-a)(1-\alpha)(1-a\alpha)}$.

G. F. for covariants, $\dfrac{1}{(1-a\alpha)(1-ax)(1-\alpha x)}$.

Groundforms:

Of order 0 . . . . (1, 1).
" " 1 . . . . (0, 1), (1, 0).

* "Linear" is here used as a noun, in conformity with the use of the words quadric, cubic, &c.

System of Linear and Quadric.

G. F. for differentiants, $\dfrac{1+ab}{(1-a)(1-b)(1-b^2)(1-a^2b)}$.

G. F. for covariants, $\dfrac{1+abx}{(1-b^2)(1-a^2b)(1-ax)(1-bx^2)}$.

Groundforms:

Of order 0 . . . . (0, 2), (2, 1).
" " 1 . . . . (1, 0), (1, 1).
" " 2 . . . . (0, 1).

System of Linear and Cubic.

G. F. for differentiants, $\dfrac{1+a^2c+(a-a^3)c^2+(1-a^2)c^3-ac^4-a^3c^5}{(1-a)(1-c)(1-c^2)(1-c^4)(1-ac)(1-a^3c)}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-ac)(1-a^3c)(1-ax)(1-cx)(1-cx^3)$.

Numerator: $1-ac+a^2c^2+\{(-1+a^2)c+(2a-a^3)c^2-a^2c^3\}x$
$+\{ac+(1-2a^2)c^2+(-a+a^3)c^3\}x^2+\{-ac^2+a^2c^3-a^3c^4\}x^3$.

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-a^3c)(1-a^2c^2)(1-ax)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+a^3c^3+\{a^2c+ac^2+(a^2-a^4)c^3\}x+\{ac+(a-a^3)c^3-a^3c^5\}x^2$
$+\{(1-a^2)c^3-a^3c^4-a^2c^5\}x^3+\{-ac^3-a^4c^6\}x^4$.

Groundforms:

Of order 0 . . . . (0, 4), (2, 2), (3, 1), (3, 3).
" " 1 . . . . (1, 0), (1, 2), (2, 1), (2, 3).
" " 2 . . . . (0, 2), (1, 1), (1, 3).
" " 3 . . . . (0, 1), (0, 3).

System of Linear and Quartic.

G. F. for differentiants, $\dfrac{1+(a+a^3)d+(a+a^2-a^5)d^2+(1-a^3-a^4)d^3+(-a^2-a^4)d^4-a^5d^5}{(1-a)(1-d)(1-d^2)^2(1-d^3)(1-a^2d)(1-a^4d)}$.

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-a^2d)(1-a^4d)(1-ax)(1-dx^2)(1-dx^4)$.

Numerator: $1-a^2d+a^4d^2+\{a^3d+(a^3-a^5)d^2\}x+\{(-1+a^2)d$
$+(2a^2-a^4)d^2-a^4d^3\}x^2+\{ad+(a-2a^3)d^2+(-a^3+a^5)d^3\}x^3$
$+\{(1-a^2)d^2-a^2d^3\}x^4+\{-ad^2+a^3d^3-a^5d^4\}x^5$.

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-a^4d)(1-a^4d^2)(1-ax)(1-dx^4)(1-d^2x^4)$.

Numerator: $1+a^6d^3+\{a^3d+a^3d^2+(a^5-a^7)d^3\}x+\{a^2d+a^2d^2+(a^4-a^6)d^3\}x^2$
$+\{ad+ad^2+(a^3-a^5)d^3\}x^3+\{(a^2-a^4)d^3-a^6d^4-a^6d^5\}x^4$
$+\{(a-a^3)d^3-a^5d^4-a^5d^5\}x^5+\{(1-a^2)d^3-a^4d^4-a^4d^5\}x^6$
$+\{-ad^3-a^7d^6\}x^7$.

Groundforms:

Of order 0	(0, 2), (0, 3), (4, 1), (4, 2), (6, 3).
" " 1	(1, 0), (3, 1), (3, 2), (5, 3).
" " 2	(2, 1), (2, 2), (4, 3).
" " 3	(1, 1), (1, 2), (3, 3).
" " 4	(0, 1), (0, 2), (2, 3).
" " 5	(1, 3).
" " 6	(0, 3).

System of Two Quadrics.

G. F. for differentiants, $\dfrac{1+b\beta}{(1-b)(1-b^2)(1-\beta)(1-\beta^2)(1-\beta b)}$.

G. F. for covariants, $\dfrac{1+b\beta x^2}{(1-b^2)(1-\beta^2)(1-b\beta)(1-bx^2)(1-\beta x^2)}$.

Groundforms:

Of order 0	(0, 2), (1, 1), (2, 0).
" " 2	(0, 1), (1, 0), (1, 1).

System of Quadric and Cubic.

G. F. for differentiants,

$$\frac{1+(2b+b^2)c+(b+b^2+b^3)c^2+c^3-b^4c^4+(-b-b^2-b^3)c^5+(-b^2-2b^3)c^6-b^4c^7}{(1-b)(1-b^2)(1-c)(1-c^2)(1-c^4)(1-bc^2)(1-b^3c^2)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-cx)(1-cx^3)$.

Numerator: $1+b^3c^4+\{(-1+b+b^2)c+(b+b^2)c^3-b^3c^5\}x$
$+\{(1+b^3)c^2+(-b-b^4)c^4\}x^2+\{bc+(-b^2-b^3)c^3$
$+(-b^2-b^3+b^4)c^5\}x^3+\{-bc^2-b^4c^6\}x^4$.

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+b^3c^4+\{(b+b^2)c+(b+b^2)c^3\}x+\{(b+b^2+b^3)c^2$
$+(b^2-b^4)c^4-b^3c^6\}x^2+\{bc+(1-b^2)c^3+(-b-b^2-b^3)c^5\}x^3$
$+\{(-b^2-b^3)c^4+(-b^2-b^3)c^6\}x^4+\{-bc^3-b^4c^7\}x^5$.

Groundforms:

Of order 0 . . . . (0, 4), (1, 2), (2, 0), (3, 2), (3, 4).
" " 1 . . . . (1, 1), (1, 3), (2, 1), (2, 3).
" " 2 . . . . (0, 2), (1, 0), (1, 2).
" " 3 . . . . (0, 1), (0, 3), (1, 1).

SYSTEM OF QUADRIC AND QUARTIC.

G. F. for differentiants, $\dfrac{1+(b+b^2)d+(2b-b^3)d^2+(1-2b^2)d^3+(-b-b^2)d^4-b^3d^5}{(1-b)(1-b^2)(1-d)(1-d^2)^2(1-d^3)(1-bd)(1-b^2d)}$.

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-bd)(1-b^2d)(1-bx^2)(1-dx^2)$
$(1-dx^4)$.

Numerator: $1-bd+b^2d^2+\{(-1+b+b^2)d+(2b-b^3)d^2-b^2d^3\}x^2$
$+\{bd+(1-2b^2)d^2+(-b-b^2+b^3)d^3\}x^4+\{-bd^2$
$+b^2d^3-b^3d^4\}x^6$.

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-b^2d)(1-b^2d^2)(1-bx^2)(1-dx^4)$
$(1-d^2x^4)$.

Numerator: $1+b^3d^3+\{(b+b^2)d+(b+b^2)d^2+(b^2-b^4)d^3\}x^2+\{bd+bd^2$
$+(b-b^3)d^3-b^3d^4-b^3d^5\}x^4+\{(1-b^2)d^3+(-b^2-b^3)d^4$
$+(-b^2-b^3)d^5\}x^6+\{-bd^3-b^4d^6\}x^8$.

Groundforms:

Of order 0 . . . . (0, 2), (0, 3), (2, 0), (2, 1), (2, 2), (3, 3).
" " 2 . . . . (1, 0), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3).
" " 4 . . . . (0, 1), (0, 2), (1, 1), (1, 2), (1, 3).
" " 6 . . . . (0, 3).

SYSTEM OF TWO CUBICS.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-\gamma)(1-\gamma^2)(1-\gamma^4)(1-c\gamma)$
$(1-c^3\gamma)(1-c\gamma^3)$.

Numerator: $1+c^3+(2c+2c^2-c^5-c^6)\gamma+(2c+2c^2-c^4-c^5-c^6-c^7)\gamma^2$
$+(1+2c^3-c^4-2c^5-c^6-c^7)\gamma^3+(-c^2-c^3-c^5-c^6)\gamma^4$
$+(-c-c^2-2c^3-c^4+2c^5+c^8)\gamma^5+(-c-c^2-c^3-c^4+2c^6+2c^7)\gamma^6$
$+(-c^2-c^3+2c^6+2c^7)\gamma^7+(c^5+c^8)\gamma^8$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-cx)(1-cx^3)$
$(1-\gamma x)(1-\gamma x^3)$.

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^0	c^0	1						
	c^2			1				
	c^3				1			
	c^5						1	
x^1	c^0		$\bar{1}$					
	c^1	$\bar{1}$		1		1		
	c^2		1					
	c^4		1					
	c^5							$\bar{1}$
	c^6						$\bar{1}$	
x^2	c^0			1				
	c^1		2				$\bar{1}$	
	c^2	1		$\bar{1}$		$\bar{1}$		
	c^4			$\bar{1}$				
	c^5		$\bar{1}$					
	c^6							1

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^6	c^2		1					
	c^4				1			
	c^5					1		
	c^7							1
x^5	c^1		$\bar{1}$					
	c^2	$\bar{1}$						
	c^3						1	
	c^5						1	
	c^6			1		1		$\bar{1}$
	c^7						$\bar{1}$	
x^4	c^1	1						
	c^2						$\bar{1}$	
	c^3					$\bar{1}$		
	c^5			$\bar{1}$		$\bar{1}$		1
	c^6		$\bar{1}$				2	
	c^7					1		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^3	c^1		$\bar{1}$		$\bar{1}$		
	c^2	$\bar{1}$				1	
	c^3				$\bar{1}$		$\bar{1}$
	c^4	$\bar{1}$		$\bar{1}$			
	c^5		1				$\bar{1}$
	c^6			$\bar{1}$		$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-c^2x^2)(1-cx^3)$
$(1-\gamma^2x^2)(1-\gamma x^3)$.

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^0	c^0	1							
	c^2			1					
	c^3				1				
	c^5						1		
x^1	c^1			1		1			
	c^2		1		1				
	c^3			1		1			
	c^4		1		1				
x^2	c^1		1		1				
	c^2			1					
	c^3		1		1				
	c^4					1			
	c^5								$\bar{1}$
	c^7						$\bar{1}$		
x^3	c^0				1				
	c^1			1		$\bar{1}$		$\bar{1}$	
	c^2		1						
	c^3	1				$\bar{2}$		$\bar{1}$	
	c^4		$\bar{1}$		$\bar{2}$				
	c^5							$\bar{1}$	
	c^6		$\bar{1}$		$\bar{1}$		$\bar{1}$		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7	γ^8
x^8	c^3			1					
	c^5					1			
	c^6						1		
	c^8								1
x^7	c^4					1		1	
	c^5				1		1		
	c^6					1		1	
	c^7				1		1		
x^6	c^1			$\bar{1}$					
	c^3	$\bar{1}$							
	c^4				1				
	c^5					1		1	
	c^6						1		
	c^7					1		1	
x^5	c^2			$\bar{1}$		$\bar{1}$		$\bar{1}$	
	c^3		$\bar{1}$						
	c^4					$\bar{2}$		$\bar{1}$	
	c^5		$\bar{1}$		$\bar{2}$				1
	c^6							1	
	c^7		$\bar{1}$		$\bar{1}$		1		
	c^8					1			

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^4	c^1	1				$\bar{1}$		
	c^2				$\bar{1}$		$\bar{1}$	
	c^3			$\bar{1}$		$\bar{2}$		$\bar{1}$
	c^4		$\bar{1}$		$\bar{2}$		$\bar{1}$	
	c^5	$\bar{1}$		$\bar{2}$		$\bar{1}$		
	c^6		$\bar{1}$		$\bar{1}$			
	c^7			$\bar{1}$				1

*Table of Groundforms.**

Order in the Variables.	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3	4
0	0					1
	1		1		1	
	2			1		
	3		1		1	
	4	1				
1	1			1		1
	2		1		1	
	3			1		
	4		1			

Order in the Variables	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3
2	0			1	
	1		1		1
	2	1		1	
	3		1		
3	0		1		1
	1	1		1	
	2		1		
	3	1			
4	1		1		

System of Cubic and Quartic.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-d)(1-d^2)^2(1-d^3)(1-c^2d)$
$(1-c^4d)(1-c^2d^3)(1-c^4d^3)$.

Numerator: $1+c^3+(3c+2c^2+2c^3+c^4-2c^5-c^6-c^7)\,d$
$+(3c+5c^2+2c^3+2c^4-3c^5-4c^6-2c^7-2c^8+c^9)\,d^2$
$+(1+3c^2+3c^3+c^4-c^5-6c^6-5c^7-4c^8+2c^{10})\,d^3$
$+(-c^2+c^3-c^4-2c^5-5c^6-6c^7-3c^8-c^9+3c^{10}+2c^{11}+c^{12})\,d^4$
$+(-2c^2-3c^3-3c^4-3c^5-2c^6-2c^7-c^8+2c^{10}+4c^{11}+3c^{12}+c^{13})\,d^5$
$+(-c^2-3c^3-4c^4-2c^5+c^7+2c^8+2c^9+3c^{10}+3c^{11}+3c^{12}+2c^{13})\,d^6$
$+(-c^3-2c^4-3c^5+c^6+3c^7+6c^8+5c^9+2c^{10}+c^{11}-c^{12}+c^{13})\,d^7$
$+(-2c^5+4c^7+5c^8+6c^9+c^{10}-c^{11}-3c^{12}-3c^{13}-c^{15})\,d^8$
$+(-c^6+2c^7+2c^8+4c^9+3c^{10}-2c^{11}-2c^{12}-5c^{13}-3c^{14})\,d^9$
$+(c^8+c^9+2c^{10}-c^{11}-2c^{12}-2c^{13}-3c^{14})\,d^{10}+(-c^{12}-c^{15})\,d^{11}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^2d)(1-c^4d)(1-c^2d^3)(1-c^4d^3)$
$(1-cx)(1-cx^3)(1-dx^2)(1-dx^4)$.

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*The forms of ord. 1, deg. 3, 4 and of ord. 1, deg. 4, 3 given by Clebsch and Gordan, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^0	c^0	1									
	c^2		$\bar{1}$								
	c^4			2	2	2	1				
	c^6			1	1		$\bar{1}$	$\bar{1}$			
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$			
	c^{10}								1		
	c^{12}									$\bar{1}$	
x^1	c^1	$\bar{1}$		1							
	c^3		3	2	1	1					
	c^5		1	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1			
	c^7			$\bar{2}$	$\bar{1}$	$\bar{1}$		1	$\bar{1}$		
	c^9				1	1	1		$\bar{2}$		
	c^{11}							$\bar{1}$	$\bar{2}$		
	c^{13}									1	
x^2	c^0		$\bar{1}$								
	c^2	1	1	3	2	1					
	c^4		$\bar{1}$		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^6		$\bar{1}$		$\bar{2}$	$\bar{2}$			1		
	c^8			1		1	1		2		
	c^{10}						$\bar{1}$	$\bar{1}$		$\bar{2}$	
	c^{12}							1	1		1
x^3	c^1		2								
	c^3			$\bar{3}$							
	c^5		$\bar{1}$	$\bar{2}$	1		1		$\bar{1}$		
	c^7					$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	1	
	c^9						$\bar{1}$			1	
	c^{11}					1	1	2	1	2	
	c^{13}									1	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
x^8	c^2		1								
	c^4			$\bar{1}$							
	c^6				2	2	2	1			
	c^8				1	1		$\bar{1}$	$\bar{1}$		
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{12}									1	
	c^{14}										$\bar{1}$
x^7	c^1		$\bar{1}$								
	c^3			2	1						
	c^5			2		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^7			1	$\bar{1}$		1	1	2		
	c^9				$\bar{1}$	1	1	1	2	$\bar{1}$	
	c^{11}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	
	c^{13}								$\bar{1}$	$\bar{1}$	1
x^6	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$						
	c^4		2		1	1					
	c^6			$\bar{2}$		$\bar{1}$	$\bar{1}$		$\bar{1}$		
	c^8			$\bar{1}$			2	2		1	
	c^{10}				1	2	2	1		1	
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{14}									1	
x^5	c^1	1	$\bar{1}$								
	c^3		$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$				
	c^5		$\bar{1}$			1					
	c^7		$\bar{1}$	1	2	1	1				
	c^9			1		$\bar{1}$		$\bar{1}$	2	1	
	c^{11}								3		
	c^{13}									$\bar{2}$	

Numerator—*Continued:*

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^4	c^0		1							
	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$	$\bar{1}$				
	c^4	$\bar{1}$		$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		1	$\bar{2}$	$\bar{1}$			1		
	c^8			$\bar{1}$			1	2	$\bar{1}$	
	c^{10}					1	2	2		1
	c^{12}					1	1	1		1
	c^{14}								$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^4d)(1-c^4d^2)(1-c^2d^3)(1-c^4d^3)$
$(1-cx^3)(1-c^2x^2)(1-dx^4)(1-d^2x^4)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^0	c^0	1											
	c^4			1	2	2	1						
	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{14}										$\bar{1}$		
x^1	c^1		1	1									
	c^3		2	3	2	1							
	c^5		1	2	3	2	1	1					
	c^9				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
	c^{11}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{13}								$\bar{1}$	$\bar{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^{11}	c^3			1									
	c^7					1	2	2	1				
	c^9					1	3	2	1				
	c^{11}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{13}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{17}												$\bar{1}$
x^{10}	c^4				1	1							
	c^6				2	3	2	1					
	c^8				1	2	3	2	1	1			
	c^{12}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	c^{14}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^{16}										$\bar{1}$	$\bar{1}$	

Numerator—*Continued:*

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^2	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}										1		
x^3	c^1		1	1	1								
	c^3	1	1	3	5	3	1						
	c^5		$\bar{1}$		2	3	2						
	c^7		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^9					$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{1}$			
	c^{11}				1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$		
	c^{13}						1	1	2	1	$\bar{1}$		
	c^{15}								1	1	1		
x^4	c^2		1	2	3	1							
	c^4		$\bar{1}$		2	1	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}					$\bar{1}$	$\bar{1}$	1	1	1			
	c^{12}					1	3	4	4	2			
	c^{14}						1	2	3	2	1	1	1
x^5	c^1		1	1									
	c^3				$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^5		$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^7			$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^9					$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	1	
	c^{11}					1	2	2	4	5	4	2	
	c^{13}						1	3	4	4	4	1	
	c^{15}							1	1	1	1		
	c^{17}										$\bar{1}$		

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^9	c^1			$\bar{1}$									
	c^5			1	2	2		$\bar{1}$	$\bar{1}$				
	c^7				2	3	2	1					
	c^9					1	2	3	3	3	1		
	c^{13}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{15}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
x^8	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4			1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$					
	c^6			1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^8				1	3	3	4	2				
	c^{10}					1	3	3	3	4	2	1	
	c^{12}							$\bar{2}$	$\bar{3}$	$\bar{2}$		1	
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{16}									$\bar{1}$	$\bar{1}$	$\bar{1}$	
x^7	c^3	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^5				$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^7				$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1				
	c^9			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{11}					1	3	4	4	4	2	1	
	c^{13}					1	2	1	$\bar{1}$	$\bar{2}$		1	
	c^{15}								$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
x^6	c^0			1									
	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4		$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$					
	c^6		$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1				
	c^{10}				1	2	2	4	5	4	2		
	c^{12}				1	2	2	3	4	5	3	1	
	c^{14}						1	1	1	1			
	c^{16}										$\bar{1}$	$\bar{1}$	

*Table of Groundforms.**

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	1		1	1			
	3		2	3	2	1	
	5		1	2	2		

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3
2	2	1	2	2	1
	4		2	2	
3	1	1	1	1	1
	3	1	1	1	1
4	0		1	1	
	2		1	1	1
5	1		1	1	
6	0				1

System of Two Quartics.

G. F. for differentiants,

Denominator: $(1-d)(1-d^2)^2(1-d^3)(1-\delta)(1-\delta^2)^2(1-\delta^3)(1-d\delta)(1-d^2\delta)$
$(1-d\delta^2)$.

Numerator: $1+d^3+(3d+3d^2-d^4-d^5)\delta+(3d+4d^2-d^3-3d^4-2d^5-d^6)\delta^2$
$+(1-d^2-2d^4-3d^5-d^6)\delta^3+(-d-3d^2-2d^3-d^5+d^7)\delta^4$
$+(-d-2d^2-3d^3-d^4+4d^5+3d^6)\delta^5+(-d^2-d^3+3d^5+3d^6)\delta^6$
$+(d^4+d^7)\delta^7$.

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^2)(1-dx^4)(1-\delta x^2)(1-\delta x^4)$.

---

* The form of ord. 1, deg. 5, 4, and the two forms of ord. 2, deg. 4, 3, given by Gundelfinger, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist.

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5
x^0	d^0	1					
	d^2			1			
	d^4					1	
x^2	d^0		$\bar{1}$				
	d^1	$\bar{1}$	1	1	1		
	d^2		1	1			
	d^3		1		1		
	d^4						$\bar{1}$
	d^5					$\bar{1}$	
x^4	d^0			1			
	d^1		2		$\bar{1}$	$\bar{1}$	
	d^2	1		$\bar{1}$	$\bar{2}$		
	d^3		$\bar{1}$	$\bar{2}$			
	d^4		$\bar{1}$				$\bar{1}$
	d^5					$\bar{1}$	1

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^{10}	d^2		1				
	d^4				1		
	d^6						1
x^8	d^1		$\bar{1}$				
	d^2	$\bar{1}$					
	d^3			1		1	
	d^4				1	1	
	d^5			1	1	1	$\bar{1}$
	d^6					$\bar{1}$	
x^6	d^1	1	$\bar{1}$				
	d^2	$\bar{1}$				$\bar{1}$	
	d^3				$\bar{2}$	$\bar{1}$	
	d^4			$\bar{2}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{1}$		2	
	d^6				1		

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^4)(1-d^2x^4)(1-\delta x^4)(1-\delta^2x^4)$.

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^0	d^0	1						
	d^2			1				
	d^4					1		
x^2	d^1		1	1	1			
	d^2		1	1	1			
	d^3		1	1	1			
x^4	d^1		1	1				
	d^2		1	1				
	d^3				1	1		
	d^4				1		$\overline{1}$	$\overline{1}$
	d^5					$\overline{1}$		
	d^6					$\overline{1}$		
x^6	d^0				1			
	d^1		1	1		$\overline{1}$	$\overline{1}$	
	d^2		1	1	$\overline{1}$	$\overline{2}$	$\overline{1}$	
	d^3	1		$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	
	d^4		$\overline{1}$	$\overline{2}$	$\overline{2}$			
	d^5		$\overline{1}$	$\overline{1}$	$\overline{1}$			

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7
x^{14}	d^3			1				
	d^5					1		
	d^7							1
x^{12}	d^4				1	1	1	
	d^5				1	1	1	
	d^6				1	1	1	
x^{10}	d^1			$\overline{1}$				
	d^2			$\overline{1}$				
	d^3	$\overline{1}$	$\overline{1}$		1			
	d^4			1	1			
	d^5					1	1	
	d^6					1	1	
x^8	d^2				$\overline{1}$	$\overline{1}$	$\overline{1}$	
	d^3				$\overline{2}$	$\overline{2}$	$\overline{1}$	
	d^4		$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{1}$		1
	d^5		$\overline{1}$	$\overline{2}$	$\overline{1}$	1	1	
	d^6		$\overline{1}$	$\overline{1}$		1	1	
	d^7				1			

*Table of Groundforms.**

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
2	1		1	1	1
	2		1	1	1
	3		1	1	

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
4	0		1	1	
	1	1	1	1	
	2	1	1		
6	0				1
	1		1	1	
	2		1		
	3	1			

The following table exhibits the total numbers of groundforms; the quantics themselves and the absolute constant are included in the numbers.†

Order of Quantic.	Order of Quantic. 0	1	2	3	4
0	1	2	3	5	6
1		4	6	14	21
2			7	16	19
3				27	62
4					29

* The forms of ord. 4, deg. 2, 2, and of ord. 6, deg. 2, 2, given by Gordan, do not appear in this table, and have been proved by the author to be compound forms.

† Some remarks on the preceding tables (to save delay in going to press) have been made the subject of a separate article in this number.

A New General Method of Interpolation.

BY EMORY MCCLINTOCK.

A GENERAL method of interpolation will be described below, which is apparently both easier to prove and easier to work than the general methods now in use. By general method, I mean a process applicable to all cases, including those in which the given values of the variable do not form an arithmetical progression. Before stating it, it will be well to recall briefly the two methods hitherto prevailing.

The first method, as we may call it for the sake of distinction, is known as that of Lagrange, though the credit of it has also been claimed for Euler. If we suppose that ϕx_1, ϕx_2, ϕx_3, $\ldots$ ϕx_k are given, where x_1, x_2, $\ldots$ x_k are any numbers, Lagrange's formula for determining ϕx, where x is any other number, is

$$\phi x = \mathrm{X}_1 \phi x_1 + \mathrm{X}_2 \phi x_2 + \mathrm{X}_3 \phi x_3 + \ldots + \mathrm{X}_k \phi x_k,$$

where

$$\mathrm{X}_1 = \frac{(x - x_2)(x - x_3)\ldots(x - x_k)}{(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_k)},$$

$$\mathrm{X}_2 = \frac{(x - x_1)(x - x_3)\ldots(x - x_k)}{(x_2 - x_1)(x_2 - x_3)\ldots(x_2 - x_k)},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{X}_k = \frac{(x - x_1)(x - x_2)\ldots(x - x_{k-1})}{(x_k - x_1)(x_k - x_2)\ldots(x_k - x_{k-1})}.$$

The law by which the coefficients are formed is obvious, and the proof of the formula is not difficult.

The second method, that of Newton, is as follows, using the same notation as before. Let

$$\phi_1 x_2 = \frac{\varphi x_2 - \varphi x_1}{x_2 - x_1},$$

$$\phi_1 x_3 = \frac{\varphi x_3 - \varphi x_2}{x_3 - x_2},$$

and so on, the general form being

$$\phi_1 x_n = \frac{\varphi x_n - \varphi x_{n-1}}{x_n - x_{n-1}}.$$

The most convenient way of recording these divided differences is to set down, first, a column containing the numerical values of x_1, x_2, &c.; next, a column containing, opposite these numbers respectively, the values of ϕx_1, ϕx_2, &c.; and thirdly, a column containing $\phi_1 x_2$ opposite x_2, $\phi_1 x_3$ opposite x_3, and so on. Adjoining this column of differences, may be recorded a second series of differences, $\phi_2 x_3$, $\phi_2 x_4$, &c., calculated by the formula

$$\phi_2 x_n = \frac{\varphi_1 x_n - \varphi_1 x_{n-1}}{x_n - x_{n-2}};$$

then a third series,

$$\phi_3 x_n = \frac{\varphi_2 x_n - \varphi_2 x_{n-1}}{x_n - x_{n-3}},$$

and so on, the general rule being

$$\phi_{m+1} x_n = \frac{\varphi_m x_n - \varphi_m x_{n-1}}{x_n - x_{n-m-1}}.$$

Thus, if four terms have been given, we shall have this scheme,

x_n	ϕx_n	$\phi_1 x_n$	$\phi_2 x_n$	$\phi_3 x_n$
x_1	ϕx_1			
x_2	ϕx_2	$\phi_1 x_2$		
x_3	ϕx_3	$\phi_1 x_3$	$\phi_2 x_3$	
x_4	ϕx_4	$\phi_1 x_4$	$\phi_2 x_4$	$\phi_3 x_4$

To find ϕx, we have now to use this formula:

$$\phi x = \phi x_1 + (x-x_1)\,\phi_1 x_2 + (x-x_1)(x-x_2)\,\phi_2 x_3 + (x-x_1)(x-x_2)(x-x_3)\,\phi_3 x_4 + \dots.$$

The complete proof of this formula is laborious.*

Concerning the relative merits of these two methods, it may be said that if more than one term is to be interpolated the second is by far the best. If only one term is wanted, the matter is doubtful. Some writers give both methods; Boole, in doing so (*Finite Diffs.*, 2d ed.), shows a preference for the first method, Grunert for the second. Some writers, possibly to save space, mention only one of the two; Hymers gives the first, De Morgan the second. The first formula has often been commended for its elegance, exhibited both in the symmetry of its form and in the simplicity of its proof. The second, on the other hand, seems to be, on the whole, the easier to work in practice. After due examination, my conclusion is that, if all the multiplications and

*See, for three different demonstrations, De Morgan, *Calculus*, pp. 550-552; Grunert, Supplement to Klügel's *Wörterbuch*, vol. II, pp. 43-46; Boucharlat, *Calcul*, 7th ed., pp. 417-420.

divisions are performed by logarithms, the first method can, adopting one or two obvious labor-saving devices, be worked with somewhat fewer references to the table of logarithms than the second, but that, on the other hand, it requires a more frequent use of some of the logarithms when found, and imposes a greater strain of attention; in short, that the second method is, for practical purposes, at least equal to the first. This, of course, is a matter which any one interested will examine and decide for himself.

The alternative method which I have to present resembles, and may be regarded, perhaps, as a substitute for, the second method. The formula is the same, the difference lying in the manner of obtaining the quantities $\phi_1 x_2$, $\phi_2 x_3$, &c. I form the several orders of divided differences according to this formula,

$$\phi_{m+1} x_n = \frac{\varphi_m x_n - \varphi_m x_{m+1}}{x_n - x_{m+1}}.$$

In applying this formula, ϕ_0 is to be considered equivalent to ϕ, the given terms being ϕx_1 or $\phi_0 x_1$, ϕx_2 or $\phi_0 x_2$, and so on. The computation of divided differences by this formula is simpler than by the second method, inasmuch as *the denominator, in all the orders, corresponds to the numerator.* The amount of labor required in both cases is the same, but the strain on the attention is lessened by employing a more symmetrical formula. The work done is more mechanical, and therefore easier. This statement may readily be tested.

The demonstration of the new method is so simple that the process might be said to prove itself. Since, from the equation last given,

$$\phi_m x_n = \phi_m x_{m+1} + (x_n - x_{m+1})\, \phi_{m+1} x_n,$$

we have successively, for $m = 0$, $m = 1$, &c.,

$$\phi_0 x_n = \phi_0 x_1 + (x_n - x_1)\, \phi_1 x_n,$$
$$\phi_1 x_n = \phi_1 x_2 + (x_n - x_2)\, \phi_2 x_n,$$
$$\phi_2 x_n = \phi_2 x_3 + (x_n - x_3)\, \phi_3 x_n,$$

and so on. Hence, by successive substitution,

$$\phi_0 x_n = \phi_0 x_1 + (x_n - x_1)\, \phi_1 x_2 + (x_n - x_1)(x_n - x_2)\, \phi_2 x_3 + (x_n - x_1)(x_n - x_2)$$
$$(x_n - x_3)\, \phi_3 x_4 + \ldots \; (k \text{ terms}).$$

This general law is true, as proved, for all the given values of x_n, namely, x_1, x_2, ... x_k. Assuming the same law to be true of values not given, it becomes a convenient formula for interpolation. This assumption is the same in substance with those underlying the first and second methods, and

the values determined in any given case by the three methods are identical. If we write x for x_n, and ϕ for ϕ_0, the formula becomes

$$\phi x = \phi x_1 + (x - x_1)\,\phi_1 x_2 + (x - x_1)(x - x_2)\,\phi_2 x_3 + \ldots .$$

It may also be written in this form:

$$\phi x = \phi x_1 + (x - x_1)[\phi_1 x_2 + (x - x_2)\{\phi_2 x_3 + (x - x_3)[\phi_3 x_4 + \ldots]\}].$$

If, for example, four terms are given, we may calculate successively

$$\psi_3 x_4 = (x - x_3)\,\phi_3 x_4,$$
$$\psi_2 x_3 = (x - x_2)\,(\phi_2 x_3 + \psi_3 x_4),$$
$$\psi_1 x_2 = (x - x_1)\,(\phi_1 x_2 + \psi_2 x_3),$$

and the general formula is reduced to this,

$$\phi x = \phi x_1 + \psi_1 x_2 .$$

Similarly, if five terms are given, we may begin by calculating

$$\psi_4 x_5 = (x - x_4)\,\phi_4 x_5 .$$

then

$$\psi_3 x_4 = (x - x_3)\,(\phi_3 x_4 + \psi_4 x_5),$$

and the rest as before. The same manner of lessening the labor may be followed for six or more given terms.

The results obtained by this method are neither more nor less accurate than those obtained by the preceding methods, being, as already stated, the same. In the following example, where seven values of a function of the sixth degree are given, the result is exactly correct.

x_n	$\phi_0 x_n$	$\phi_1 x_n$	$\phi_2 x_n$	$\phi_3 x_n$	$\phi_4 x_n$	$\phi_5 x_n$	$\phi_6 x_n$
$x_1 = 0$	5						
$x_2 = 1$	1	-4					
$x_3 = 4$	1	-1	1				
$x_4 = 5$	3	$-\frac{2}{5}$	$\frac{9}{10}$	$-\frac{1}{10}$			
$x_5 = 6$	5	0	$\frac{4}{5}$	$-\frac{1}{10}$	0		
$x_6 = 8$	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{120}$	$-\frac{1}{240}$	
$x_7 = 11$	-6	-1	$\frac{3}{10}$	$-\frac{1}{10}$	0	0	$\frac{1}{720}$

Each difference is derived from the one adjoining on the left, by subtracting from the latter the number at the top of the same column, and dividing the remainder by the corresponding difference taken from the first column. Let us suppose that $x=2$; then, following the formulæ last indicated, we have, successively,

$$\begin{aligned}
&\psi_6 x_7 = (2-8)\,\frac{1}{720} = -\frac{1}{120}\,,\\
&\psi_5 x_6 = (2-6)\left(-\frac{1}{240}-\frac{1}{120}\right) = \frac{1}{20}\,,\\
&\psi_4 x_5 = (2-5)\left(0+\frac{1}{20}\right) = -\frac{3}{20}\,,\\
&\psi_3 x_4 = (2-4)\left(-\frac{1}{10}-\frac{3}{20}\right) = \frac{1}{2}\,,\\
&\psi_2 x_3 = (2-1)\left(1+\frac{1}{2}\right) = \frac{3}{2}\,,\\
&\psi_1 x_2 = (2-0)\left(-4+\frac{3}{2}\right) = -5\,,\\
&\phi x = 5-5 = 0\,.
\end{aligned}$$

Note on the Demonstration of certain Formulæ for Interpolation.

When the given values of the variable are in arithmetical progression, interpolation is best performed by one of two well known formulæ, the choice between the two depending upon the circumstances of the case in hand. These formulæ are,

$$\phi\,(x+k) = \phi x + k\Delta\phi x + \frac{1}{2}\,k\,(k-1)\,\Delta^2\phi x + \ldots,$$

$$\phi\,(x+k) = \phi x + k\left[\Lambda\phi\left(x\pm\frac{1}{2}\right) + \frac{1}{2}\,(k\mp 1)\,\Lambda^2\phi x\right] + \frac{1}{2.3}\,k^{)3(}\left[\Lambda^3\phi\left(x\pm\frac{1}{2}\right) + \frac{1}{4}\,(k\mp 2)\,\Lambda^4\phi x\right] + \ldots.$$

The notation here used is that employed in my "Essay on the Calculus of Enlargement" (Amer. Journal, II, 101), namely,

$$k^{)3(} = (k+1)\,k\,(k-1)\,,$$
$$k^{)5(} = (k+2)(k+1)\,k\,(k-1)(k-2)\,,$$

and so on; also,

$$\Delta = \mathrm{E} - 1\,,$$
$$\Lambda = \mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\,,$$

where E is the symbol of Enlargement, E^h being an operation such that $E^h\phi x = \phi\,(x+h)$, whence

$$\Delta\phi x = \phi\,(x+1) - \phi x\,,$$

$$\Lambda\phi x = \phi\left(x+\frac{1}{2}\right) - \phi\left(x-\frac{1}{2}\right),$$

$$\Lambda^2\phi x = \Lambda\phi\left(x+\frac{1}{2}\right) - \Lambda\phi\left(x-\frac{1}{2}\right),$$

and so on. The class of differences denoted by $\Lambda^n\phi x$ are called central differences. Of the two formulæ above stated, the first is a simple and obvious case of the Factorial Theorem given in the Essay referred to, p. 143. The second may be derived from the same theorem, as follows.

From the factorial theorem we have directly

$$\phi E = \phi 1 + \phi E 0 . \Lambda + \frac{1}{2}\,\phi E 0^{(2)} . \Lambda^2 + \frac{1}{2.3}\,\phi E 0^{(3)} . \Lambda^3 + \ldots, \qquad \text{(A)}$$

where $x^{(2)} = xx$, $x^{(3)} = x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$, $x^{(4)} = x\,(x+1)\,x\,(x-1)$, $x^{(5)} = x\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$, and so on, the general form being

$$x^{(m)} = x\left(x+\frac{1}{2}\,m-1\right)\left(x+\frac{1}{2}\,m-2\right)\ldots\left(x-\frac{1}{2}\,m+1\right).$$

Also, in terms of mean central differences,

$$\phi E = \phi 1 . I + \phi E 0 . I\Lambda + \frac{1}{2}\,\phi E 0^{)2(} . I\Lambda^2 + \frac{1}{2.3}\,\phi E 0^{)3(} . I\Lambda^3 + \ldots. \qquad \text{(B)}$$

Here,

$$I = \frac{1}{2}\,(E^{\frac{1}{2}} + E^{-\frac{1}{2}}),$$

$$I\Lambda^n\phi x = \frac{1}{2}\left(\Lambda^n\phi\left[x+\frac{1}{2}\right] + \Lambda^n\phi\left[x-\frac{1}{2}\right]\right);$$

also,

$$x^{)m(} = \left(x+\frac{1}{2}\,m-\frac{1}{2}\right)\left(x+\frac{1}{2}\,m-\frac{3}{2}\right)\ldots\left(x-\frac{1}{2}\,m+\frac{1}{2}\right).$$

Hence,

$$\Lambda = \Lambda 0 . I\Lambda + \frac{1}{2}\,\Lambda 0^{)2(} . I\Lambda^2 + \frac{1}{2.3}\,\Lambda 0^{)3(} . I\Lambda^3 + \ldots.$$

Since $\Lambda x^{)m(} = m x^{)m-1(}$, as was shown in the Essay referred to,

$$\Lambda = I\Lambda + 0^{)1(} . I\Lambda^2 + \frac{1}{2}\,0^{)2(} . I\Lambda^3 + \frac{1}{2.3}\,0^{)3(} I\Lambda^4 + \ldots.$$

But $0^{)1(} = 0$, $0^{)3(} = 0$, and so for all odd exponents; so that

$$\Lambda = \mathrm{I}\Lambda + \frac{1}{2}\,0^{)2(}.\,\mathrm{I}\Lambda^3 + \frac{1}{2.3.4}\,0^{)4(}.\,\mathrm{I}\Lambda^5 + \ldots,$$

and multiplying this by Λ^{n-1},

$$\Lambda^n = \mathrm{I}\Lambda^n + \frac{1}{2}\,0^{)2(}.\,\mathrm{I}\Lambda^{n+2} + \frac{1}{2.3.4}\,0^{)4(}.\,\mathrm{I}\Lambda^{n+4} + \ldots.$$

It follows that any even or odd central difference, or any expression composed of even or odd central differences, can be expressed, respectively, in even or odd mean central differences. Let

$$\phi_1\mathrm{E} = \phi 1 + \frac{1}{2}\,\phi\mathrm{E}0^{(2)}.\,\Lambda^2 + \frac{1}{2.3.4}\,\phi\mathrm{E}0^{(4)}.\,\Lambda^4 + \ldots,$$

$$\phi_2\mathrm{E} = \phi\mathrm{E}0\,.\,\Lambda + \frac{1}{2.3}\,\phi\mathrm{E}0^{(3)}.\,\Lambda^3 + \ldots,$$

so that, observing equation (A), $\phi_1\mathrm{E} + \phi_2\mathrm{E} = \phi\mathrm{E}$. Then

$$\phi_1\mathrm{E} = a_0\mathrm{I} + a_2\mathrm{I}\Lambda^2 + a_4\mathrm{I}\Lambda^4 + \ldots,$$

$$\phi_2\mathrm{E} = a_1\mathrm{I}\Lambda + a_3\mathrm{I}\Lambda^3 + \ldots,$$

where a_0, a_2, &c., a_1, a_3, &c., are undetermined coefficients, and

$$\phi\mathrm{E} = a_0\mathrm{I} + a_1\mathrm{I}\Lambda + a_2\mathrm{I}\Lambda^2 + \ldots.$$

But, by (B),

$$\phi\mathrm{E} = \phi 1\,.\,\mathrm{I} + \phi\mathrm{E}0\,.\,\mathrm{I}\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{)2(}.\,\mathrm{I}\Lambda^2 + \ldots.$$

This determines the values of the coefficients, and it follows that the even or odd terms of (A) may be replaced, respectively, by the even or odd terms of (B). We derive in this way the following novel and important theorems in the Calculus of Enlargement, expressing $\phi\mathrm{E}$ in alternate central and mean central differences:

$$\phi\mathrm{E} = \phi 1 + \phi\mathrm{E}0\,.\,\mathrm{I}\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{(2)}.\,\Lambda^2 + \frac{1}{2.3}\,\phi\mathrm{E}0^{)3(}.\,\mathrm{I}\Lambda^3 + \frac{1}{2.3.4}\,\phi\mathrm{E}0^{(4)}.\,\Lambda^4 + \ldots,$$

$$\phi\mathrm{E} = \phi 1\,.\,\mathrm{I} + \phi\mathrm{E}0\,.\,\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{)2(}.\,\mathrm{I}\Lambda^2 + \frac{1}{2.3}\,\phi\mathrm{E}0^{(3)}.\,\Lambda^3 + \frac{1}{2.3.4}\,\phi\mathrm{E}0^{)4(}.\,\mathrm{I}\Lambda^4 + \ldots.$$

These general theorems can, among other uses, be applied directly to the matter now in hand, interpolation. Let $\phi\mathrm{E} = \mathrm{E}^k$, and let ψx be the subject of operation; then

$$\psi(x+k) = \psi x + k\,\mathrm{I}\Lambda\psi x + \frac{1}{2}\,k^{(2)}\Lambda^2\psi x + \frac{1}{2.3}\,k^{)3(}\mathrm{I}\Lambda^3\psi x + \ldots,$$

$$\psi(x+k) = \mathrm{I}\psi x + k\Lambda\psi x + \frac{1}{2}\,k^{)2(}\mathrm{I}\Lambda^2\psi x + \frac{1}{2.3}\,k^{(3)}\Lambda^3\psi x + \ldots,$$

These formulæ are usually ascribed to Stirling, who published them in 1730. I have, however, seen the first and more important of the two quoted from Newton, 1711. They remained undemonstrated, it is said, until the discovery of the method of generating functions. The proof by that method, which may be found in Lacroix, is tedious and indirect, yet it is the only one I have met with, though other indirect demonstrations are certainly possible.*

Since $\mathrm{I} = \mathrm{E}^{\pm\frac{1}{2}} \mp \frac{1}{2}\Delta$,

$$\mathrm{I}\Delta^n\psi x = \Delta^n\psi\left(x \pm \frac{1}{2}\right) \mp \frac{1}{2}\Delta^{n+1}\psi x.$$

Making this substitution in the first formula just demonstrated, and observing that $k^{(n)} = k\,.\,k^{)n-1(}$, we derive the duplex formula which we set out to prove,

$$\psi(x+k) = \psi x + k\left[\Delta\psi\left(x \pm \frac{1}{2}\right) + \frac{1}{2}(k \mp 1)\Delta^2\psi x\right]$$
$$+ \frac{1}{2\,.\,3}k^{)3(}\left[\Delta^3\psi\left(x \pm \frac{1}{2}\right) + \frac{1}{4}(k \mp 2)\Delta^4\psi x\right] + \ldots.$$

* The difficulty does not lie in exhibiting the first few coefficients, but in proving their law. The first of these formulæ is said by Boole to have been discussed for Smith's Prize in 1860.

A Certain Class of Cubic Surfaces treated by Quaternions.

By A. B. Chace, *Valley Falls, R. I.*

The general scalar equation of the third degree in an unknown vector ρ may be written in the form

$$H + Sa_1\rho b_1 + Sa\rho b\rho c + Sl'\rho m\rho n\rho p = 0,$$

where H is a known scalar quantity and a_1, b_1, a, b, &c., are known quaternions. But we have

1st. $Sa_1\rho b_1 = S(Sa_1 + Va_1)\,\rho\,(Sb_1 + Vb_1) = Sa_1 \,.\, S\rho\, Vb_1 + Sb_1 \,.\, S\rho\, Va_1 + S\,.\,\rho\, Vb_1\, Va_1$.

This may evidently be written in the form $S\delta\rho$, where δ is a known vector.

2d. $Sa\rho b\rho c = Sca\rho b\rho = Sa'\rho b\rho = (2Sa'Sb - Sa'b)\,\rho^2 + 2Sa'\rho Sb\rho$.

This may be written $E\rho^2 + 2S\alpha\rho S\beta\rho$, where E is a known scalar quantity and α and β known vectors.

3d. $Sl'\rho m\rho n\rho p = Spl'\rho m\rho n\rho = Sl\rho m\rho n\rho = Sl\rho m\rho\,(S\nu\rho + \rho Sn + V\nu\rho)$

$$= \rho^2 S\rho\,[\lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu) - Sn\,.\,V\lambda\mu$$
$$- Sl\,.\,V\mu\nu - Sm\,.\,V\nu\lambda] + 4S\lambda\rho S\mu\rho S\nu\rho,$$

where λ, μ and ν are respectively equal to Vl, Vm and Vn.

This quantity may be written in the form $F\rho^2 S\eta\rho + 4S\lambda\rho S\mu\rho S\nu\rho$, where

$$F\eta = \lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu)$$
$$- Sl\,.\,V\mu\nu - Sm\,.\,V\nu\lambda - Sn\,.\,V\lambda\mu. \qquad \text{No. 1.}$$

Hence the general equation of the third degree may be written

$$H + S\delta\rho + E\rho^2 + 2S\alpha\rho S\beta\rho + F\rho^2 S\eta\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 0.$$

If now we assume such relations of the constants and such an origin that the terms containing the first and second powers of ρ disappear for all values of ρ, though I have as yet been unable to discover any general method of classification, the above equation becomes

$$D + F\rho^2 S\eta\rho + 4S\lambda\rho Sm\rho S\nu\rho = 0, \qquad \text{No. 2.}$$

in which it must be remembered that η is a vector dependent upon the other constants. It may however be written in terms of the three known vectors λ, μ and ν in which case we have

$$D + A\rho^2 S\lambda\rho + B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 0. \qquad \text{No. 3.}$$

This is the general equation of a class of surfaces of the third degree, which from analogy I call Central Cubics, whose peculiarities I propose to discuss somewhat briefly in the following pages.

Let us now define $\phi\rho f\rho$ as a vector function of ρ of the second degree. In case we have $S\lambda\phi\rho f\rho = S\rho\phi\lambda f\rho = S\rho\phi\rho f\lambda$ where λ is any vector, we may say that the function is self-conjugate. If however we have $S\lambda\phi\rho f\rho$ equal to $S\rho\phi\lambda f\rho$, but not to $S\rho\phi\rho fl$ we may say that the function is semi-self-conjugate. Having premised so much, let us assume a semi-self-conjugate vector function of ρ of the second degree

$$\phi\rho f\rho = A\rho S\lambda\rho + B\rho S\mu\rho + C\rho S\nu\rho + \frac{4}{3}\left[\lambda S\mu\rho S\nu\rho + \mu S\nu\rho S\lambda\rho + \nu S\lambda\rho S\mu\rho\right],$$

we may now write equation No. 3, $S\rho\phi\rho f\rho = D$, changing for convenience the sign of D. If D becomes equal to zero, we have

$$S\rho\phi\rho f\rho = 0, \qquad \text{No. 4.}$$

which is the equation of a cubical cone, and will be considered hereafter.

In case D is not zero, we may divide the constants in the above equation so that we may write

$$S\rho\phi\rho f\rho = 1, \qquad \text{No. 5.}$$

as the general equation of central surfaces of the third degree excluding cones.

If we differentiate No. 5 we have

$$Sd\rho\phi\rho f\rho + S\rho\phi d\rho f\rho + S\rho\phi\rho fd\rho = 0,$$

or, as the function is semi-self-conjugate, we may write

$$2Sd\rho\phi\rho f\rho + S\rho\phi\rho fd\rho = 0.$$

But $d\rho$ is in the direction of the variation of ρ at any instant. It is then in the direction of the tangent at the extremity of ρ. Now, if we consider ρ fixed and allow $d\rho$ to vary, we may write $(\omega-\rho)$, for $d\rho$ and the equation becomes, after reducing, $2S\omega\phi\rho f\rho + S\rho\phi\rho f\omega = 3$. This is the equation of the tangent plane.

If this plane pass through the point a, whose vector is α, we have $2S\alpha\phi\rho f\rho + S\rho\phi\rho f\alpha = 3$. This is the equation of the surface of the second degree, made by the contact of all possible tangent planes that pass through the point a, and may be called the polar quadric of the given point.

If we write the equation of the central cubic in the cyclic form we have

$$A\rho^2 S\lambda\rho + B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1,$$

where the vectors λ, μ and ν may be any vectors not inconsistent with the original supposition, by which I got rid of the first and second powers of ρ. It is, however, probable that these vectors are not restricted by this supposition, and in the rest of this discussion I have assumed them thus unrestricted. Of course such assumption is not warranted by strict mathematical logic, but the results seem so interesting and so consistent with other known mathematical truths, that I have felt myself warranted in my assumption, while waiting for the solution of what is apparently a very difficult problem, viz: the complete determination of ρ in the general vector function of the second degree. The very form of the equation proves to us the existence of three cyclic planes perpendicular to the three vectors λ, μ and ν.

To find the Radius of Curvature of any Normal Section of the Surface.

Differentiating equation No. 6, we have, assuming s as the independent variable,

$$S(2A\rho S\lambda\rho + A\rho^2\lambda + 2B\rho S\mu\rho + B\rho^2\mu + 2C\rho S\nu\rho + C\rho^2\nu + 4\lambda S\mu\rho S\nu\rho + 4\mu S\nu\rho S\lambda\rho + 4\nu S\lambda\rho S\mu\rho)\,\rho' = 0,$$

where ρ' is the first differential coefficient of ρ with reference to s, its tensor, which can be any quantity, being assumed to be one.

Taking the second differential, we have

$$S\left[\frac{2Ad\rho S\lambda\rho + 2A\lambda S\rho d\rho + 2Bd\rho S\mu\rho + 2B\mu S\rho d\rho + 2Cd\rho S\nu\rho + 2C\nu S\rho d\rho + 4\mu S\nu d\rho S\lambda\rho + 4\mu S\nu\rho S\lambda d\rho + 4\nu S\lambda d\rho S\mu\rho + 4\nu S\lambda\rho S\mu d\rho + 4\lambda S\mu d\rho S\nu\rho + 4\lambda S\mu\rho S\nu d\rho}{ds}\right]\rho' + S\beta\rho'' = 0,$$

where β is put for convenience for the normal vector. But ρ'' is perpendicular to β, hence we have, after some reduction, the following equation to determine the radius of curvature—

$$T\rho'' = \frac{1}{\beta}\,S\left[\frac{2Ad\rho S\lambda\rho + 2A\lambda S\rho d\rho + 2Bd\rho S\mu\rho + 2B\mu S\rho d\rho + 2Cd\rho S\nu\rho + 2C\nu S\rho d\rho + 4\lambda S\mu d\rho S\nu\rho + 4\lambda S\mu\rho S\nu d\rho + 4\mu S\nu d\rho S\lambda\rho + 4\mu S\nu\rho S\lambda d\rho + 4\nu S\lambda d\rho S\mu\rho + 4\nu S\lambda\rho S\mu d\rho}{d\rho}\right],$$

where $T\rho''$ is the reciprocal of the radius of absolute curvature. But we know that $d\rho$ is a vector in the tangent plane, and because the above expression is independent of $Td\rho$, we can assume any vector t, whose tensor is equal to

one that lies in the tangent plane, to be equal to $d\rho$. Substituting this, we have

$$T\rho'' = -\frac{1}{\beta} S\rho[2A\lambda + 2AtS\lambda t + 2B\mu + 2BtS\mu t + 2C\nu + 2CtS\nu t + 8\mu S\lambda tS\nu t + 8\nu S\lambda tS\mu t + 8\lambda S\mu tS\nu t],$$

which is the reciprocal of the radius of absolute curvature of any normal section in the direction of t.

If the right hand side of the equation reduces to zero for any given value of ρ, there is either a point of double inflexion, or the tensor of ρ becoming infinite, the point is at infinity, where the asymptotic line is tangent to the curve.

If now we assume t to be a vector in a plane passing through the origin perpendicular to λ, ρ will also be a vector in that plane; and the right hand member of the above equation becomes

$$-\frac{1}{\beta} S\rho\,(2B\mu + 2BtS\mu t + 2C\nu + 2CtS\nu t).$$

In order that this quantity may be equal to zero, ρ must be perpendicular to the vector

$$2B\mu + 2C\nu + t\,(2BS\mu t + 2CS\nu t),$$

and then represents probably the point of double inflation of the curvature of the section. Or we may suppose t to be an asymptotic line; in this case, ρ and t become parallel to each other, when, if $S\,(\mu 2B + \nu 2C)\,\rho = 0$, then $S\,(\mu 2B + \nu 2C)\,t = 0$, and the whole quantity becomes equal to zero. Hence, at that point, as befere, the radius of curvature is infinite. These properties of the curve will be studied more in detail hereafter, and are only referred to at present as illustrations of the use of quaternions in studying the radius of curvature.

If, in equation No. 6, the constant term disappears, the equation represents a cubical cone, as the tensor of ρ may be any quantity. If now we suppose this surface to be cut by the sphere of $\rho^2 = -1$, we have, as the equation of the Spherical Conic of the third degree,

$$4S\lambda\rho S\mu\rho S\nu\rho = AS\lambda\rho + BS\mu\rho + CS\nu\rho,$$

in which $T\rho = 1$. Hence to construct a spherical conic we have the following rule: Assuming any three points on a sphere not in the same plane, then if the continued product of the cosines of the angles which these points make with the variable point, is equal to the sum of the cosines of the angles that

the variable point makes with the three given points each multiplied by some constant quantity, then the locus of the variable point is a spherical cubic conic.

If the general surface is cut by one of the cyclic planes, as for example $S\lambda\rho = 0$, we have a cubic curve whose equation may be written

$$\rho^2 S(B\mu + C\nu)\rho = 1.$$

This curve may be called by analogy a cubical circle. It has for an asymptotic line, a vector perpendicular to $B\mu + C\nu$.

If we cut the surface by the plane $S(B\mu + C\nu)\rho = 0$, we have

$$A\rho^2 S\lambda\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1,$$

a curve which has for an asymptotic line a vector perpendicular to λ. This equation represents a curve, that may be called a cubical ellipse.

If we cut the surface by the plane $S(A\lambda + B\mu + C\nu)\rho = 0$, we have $S\lambda\rho S\mu\rho S\nu\rho = 1$, omitting the 4 for convenience. This is a curve which has asymptotic lines in three directions perpendicular respectively to λ, μ and ν, and may be called the cubical hyperbola.

Thus far we have considered the central surface only under its most general form. If now we consider certain relations to exist between the given constants, several families of surfaces readily suggest themselves.

First. If in equation No. 6 we make λ, μ and ν parallel to each other, the equation may be written

$$D\rho^2 S\lambda\rho + S^3\lambda\rho = C, \qquad \text{No. 7.}$$

where λ is a unit vector. This surface has as its limit the plane $S\lambda\rho = 0$; and as its central asymptotic line the vector $-\lambda$ prolonged, in either of which directions $T\rho = \infty$.

If we cut this surface by any plane $S\lambda\rho = -b$, we have $T\rho = \sqrt{\frac{b^3 + C}{bD}}$, the equation of a circle. Equation No. 7 represents a surface of revolution around the vector λ, and may perhaps be called a cubic cylinder.

To find that transverse plane which shall so cut the surface as to render $T\rho$ a minimum we have, considering b the independent variable and taking the first differential,

$$\frac{dT\rho}{db} = \frac{2Db^3 - DC}{2b^2D^2} \times \sqrt{\frac{bD}{b^3 + C}}.$$

In order that $T\rho$ should be a minimum, the equation must be equal to zero, which gives, as the condition required,

$$b = \sqrt[3]{\frac{C}{2}}.$$

If, in equation No. 7, we suppose D to become equal to one, its relation to an ordinary circular cylinder becomes at once apparent. We have then

$$\rho^2 S\lambda\rho + S^3\lambda\rho = C.$$

This may be changed, according to a well-known formula of quaternions, into $\sqrt{-S\lambda\rho}\,.\,TV\lambda\rho = \sqrt{C}$, and its relation to the circular cylinder becomes at once apparent, the equation of the latter being $TV\lambda\rho = A$.

Second. If, in equation No. 1, we make Sl, Sm and Sn equal to zero, and ν a vector parallel to $V\lambda\mu$, the general equation reduces to the comparatively simple form

$$A\rho^2 S\nu\rho + S\lambda\rho S\mu\rho S\nu\rho = C.$$

This equation may be written $S\rho\phi\rho S\nu\rho = C$, where $\phi\rho$ is a self-conjugate linear and vector function of ρ equal to $A\rho + \frac{1}{2}(\lambda S\mu\rho + \mu S\lambda\rho)$. This equation represents a family of surfaces of the third degree very closely related to the quadric surfaces. If C becomes zero, the equation is satisfied by any vector in the plane $S\nu\rho = 0$, and in the cone $S\rho\phi\rho = 0$. Hence it degenerates into a quadric cone and a plane at the apex. If C is not zero, we may divide the constants, so as to write the equation

$$S\rho\phi\rho S\nu\rho = 1, \qquad \text{No. 8.}$$

which is the general equation of ellipsoids and hyperboloids of this family of surfaces. Writing this equation in what Hamilton calls the rectangular form we have $cS^2 i\rho Sj\rho + c_1 S^3 j\rho + c_{11} S^2 k\rho Sj\rho = 1$, which is either an ellipsoid or hyperboloid according as c, c_1 and c_{11} are all of the same sign, or part positive and part negative. It ought perhaps to be noticed that in case the signs of all the terms are negative, the surface does not become imaginary as in the case of the corresponding quadric, but is a real surface inasmuch as the cube root of a negative quantity is a real quantity.

If we assume the order of inequality $c < c_1 < c_{11}$ and make c_1 equal to zero, the equation becomes

$$cS^2 i\rho Sj\rho + c_{11} S^2 k\rho Sj\rho = 1, \qquad \text{No. 9.}$$

the equation of a hyperbolic cubic cylinder, as c is necessarily a negative quantity. This surface is cut by any plane $Sj\rho = b$ in an ordinary hyperbola.

Now, it can be easily shown that the assumption that c_1 equals zero in this family of surfaces, obliges λ and μ, if real vectors, to be at right angles to each other. This being the case, if we wish to change equation No. 9 to the cyclic form, we may write the very simple equation $S\lambda\rho S\mu\rho S\nu\rho = 1$ as that of a particular kind of a cubic hyperbolic cylinder in which λ, μ and ν are mutually perpendicular. If we wish to express this surface in Cartesian co-ordinates we may assume, as the axes of x, y and z, lines parallel respectively to λ, μ and ν and write the very simple form $xyz = 1$.

This surface is only a particular case of the general cubic hyperbolic cylinder, in which the directions of λ, μ and ν are unlimited, the derivation of which will be shown hereafter.

Third. The fact that this family of surfaces may be expressed in terms of a variable subsidiary quadric and variable parallel planes, suggests that a similar principle may be applied to the general central cubic surface. Thus we may write the latter as follows:

$$\left(A\rho^2 + \frac{4}{3} S\mu\rho S\nu\rho\right) S\lambda\rho + \left(B\rho^2 + \frac{4}{3} S\nu\rho S\lambda\rho\right) S\mu\rho + \left(C\rho^2 + \frac{4}{3} S\lambda\rho S\mu\rho\right) S\nu\rho = 1 .$$

If we put, for the quantities in parenthesis, $S\rho\phi\rho$, $S\rho\chi\rho$ and $S\rho\psi\rho$, where $\phi\rho$, $\chi\rho$ and $\psi\rho$ are three mutually connected linear and vector self-conjugate functions of ρ, the equation becomes

$$S\rho\phi\rho S\lambda\rho + S\rho\chi\rho S\mu\rho + S\rho\psi\rho S\nu\rho = 1 ,$$

and the dependence of the general cubic surface on three connected subsidiary quadrics becomes at once apparent. If now we assume three sets of rectangular vectors, i, j, k; i_1, j_1, k_1, and i_{11}, j_{11}, k_{11}, we may write the equation

$$(aS^2 i\rho + a_1 S^2 j\rho + a_{11} S^2 k\rho)\, S\lambda\rho + (bS^2 i_1\rho + b_1 S^2 j_1\rho + b_{11} S^2 k_1\rho)\, S\mu\rho$$
$$+ (cS^2 i_{11}\rho + c_1 S^2 j_{11}\rho + c_{11} 8^2 k_{11}\rho)\, S\nu\rho = 1. \qquad \text{No. 10.}$$

In this equation we have the following equalities:

$$\begin{array}{llllll}
i = & U(\mu T\nu - \nu T\mu) & i_1 = & U(\lambda T\nu - \nu T\lambda) & i_{11} = & U(\lambda T\mu - \mu T\lambda , \\
j = & UV\mu\nu , & j_1 = & UV\lambda\nu , & j_{11} = & UV\lambda\mu , \\
k = & U(\mu T\nu + \nu T\mu) , & k_1 = & U(\lambda T\nu + \nu T\lambda) , & k_{11} = & U(\lambda T\mu + \mu T\lambda) , \\
a = & -(A + S\mu\nu) - T\mu\nu , & b = & -(B + S\lambda\nu) - T\lambda\nu , & c = & -(C + S\lambda\mu) - T\lambda\mu , \\
a_1 = & -(A + S\mu\nu) + S\mu\nu , & b_1 = & -(B + S\lambda\nu) + S\lambda\nu , & c_1 = & -(C + S\lambda\mu) + S\lambda\mu , \\
a_{11} = & -(A + S\mu\nu) + T\mu\nu , & b_{11} = & -(B + S\lambda\nu) + T\lambda\nu , & c_{11} = & -(C + S\lambda\mu) + T\lambda\mu ,
\end{array}$$

where we assume $a < a_1 < a_{11}$, $b < b_1 < b_{11}$ and $c < c_1 < c_{11}$.

This surface may be considered as one, two or threefold ellipsoidal, or one, two or threefold hyperboloidal, according as the signs of each term in one, two or all of the subsidiary quadrics are the same or different.

At this point it becomes necessary to express A, B and C of equation No. 3 in terms of Sl, Sm, Sn and $S\lambda\mu$, $S\mu\nu$ and $S\nu\lambda$ of the general equation. We have as before

$$\begin{aligned}F\eta &= \lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu) \\ &\qquad - Sl\,.\,V\mu\nu - Sm\,.\,V\nu\lambda - Sn\,.\,V\lambda\mu \\ &= \lambda\left[+SmSn - S\mu\nu + \frac{Sn\,(S\lambda\mu S\mu\nu - \mu^2 S\nu\lambda) + Sm\,(S\mu\nu S\nu\lambda - \nu^2 S\mu\lambda) + Sl\,(\mu^2\nu^2 - S^2\mu\nu)}{S\lambda\mu\nu}\right] \\ &\quad + \mu\left[SnSl - S\nu\lambda + \frac{Sn\,(S\nu\lambda S\mu\nu - \lambda^2 S\mu\nu) + Sm\,(\nu^2\lambda^2 - S^2\nu\lambda) + Sl\,(S\mu\nu S\nu\lambda - \nu^2 S\lambda\mu)}{S\lambda\mu\nu}\right] \\ &\quad + \nu\left[SlSm - S\lambda\mu + \frac{Sn\,(\lambda^2\mu^2 - S^2\lambda\mu) + Sm\,(S\mu\nu S\nu\lambda - \lambda^2 S\mu\nu) - Sl\,(S\lambda\mu S\mu\nu - \mu^2 S\nu\lambda)}{S\lambda\mu\nu}\right]\end{aligned}$$

Here the quantities in brackets are respectively equal to A, B and C of equation No. 3. Now it is evident that whatever are the vectors λ, μ, ν we can assume such a value of Sm as will render A equal to zero. In this case a_1 in equation No. 10 becomes zero, and the surface is a onefold cubic hyperbolic cylinder. In the cyclical form its equation would be

$$B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1. \qquad \text{No. 11.}$$

Again having assumed such a value of Sm as will render A equal to zero, we can easily determine a value of Sn which will render B also equal to zero. In this case a_1 and b_1 both become zero, and the surface is a twofold hyperbolic cubic cylinder and its equation in the cyclic form would be

$$C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1. \qquad \text{No. 12.}$$

Lastly, if we can discover real values for Sl, Sm and Sn such that A, B and C all become equal to zero, a_1, b_1 and c_1 are all zero and the equation represents a threefold hyperbolic cubic cylinder. To do this let us put for convenience

$$\begin{aligned} S\lambda\mu &= -a, & S\lambda\mu\nu &= -d, & T\nu^2 &= C, & Sn &= z, \\ S\mu\nu &= -b, & T\lambda^2 &= A, & Sl &= x, \\ S\nu\lambda &= -c, & T\mu^2 &= B, & Sm &= y. \end{aligned}$$

We now have the following equations in which to determine x, y and z.

$$\begin{aligned} -bd - dyz + zab - zBc + ybc - yCa - xb^2 + xBC &= 0, \\ -cd - dzx + zbc - zAb - yc^2 + yCA + xbc - xCa &= 0, \\ -ad - dxy - za^2 + zAB + ybc - yAb + xab - xBc &= 0. \end{aligned}$$

Combining these three equations, we discover single definite values for x, y and z. Hence we have single definite values of Sl, Sm and Sn, which will render A, B and C of equation No. 3 equal to zero, and the problem is solved. The cyclic equation of this surface may be written $S\lambda\mu S\mu\rho S\nu\rho = 1$, where λ, μ and ν may be any non-parallel vectors.

In the preceding pages I have attempted to classify very briefly some of the various families of the central cubic surface, and have suggested only just enough of their properties to enable one partially to conceive of their several shapes, using only the more simple of the quaternion methods. If I continue the subject, I will endeavor, in another paper, to examine more carefully a few of the surfaces here enumerated.

NOTE.—In naming the surfaces described in the preceding pages, it has seemed best to me to follow their algebraical, rather than their geometrical, analogies with the quadric surfaces.

December 22, 1879.

Remarks on the Tables for Binary Quantics in a preceding article.

BY J. J. SYLVESTER.

THE valuable idea of using different roman letters, a, b, c, d, to correspond to the coefficients of quantics of different orders, is due to Mr. Franklin. Had it occurred previously I should have employed it in the tables of the generating functions and groundforms of single quantics. The n^{th} letter of the alphabet, say θ, will in this way symbolize the $(n+1)$ coefficients $\theta_0, \theta_1, \theta_2, \ldots \theta_n$ and so x regarded as a new point of departure in the alphabet will symbolize x_0, x_1.

I pass on to a remark of greater importance referring to the separation of the Parallelopiped which may be imagined to represent the complete tabulation of the representative G. F. to a system of two simultaneous quantics, and its use in simplifying the process of tamisage.

To fix the ideas, let us take the case of a Cubic and Quartic. Then, to represent the collected signification of the rectangles at pp. 301, 302,* we may suppose a parallelopiped 12 inches in length, 17 in breadth, and 11 in depth, 12, 17, 11 being the highest exponents which appear in such rectangles of d, c, x, respectively, and confine our attention to the sign proper to each of the $12 \cdot 17 \cdot 11$ cubical spaces (inch cubes) which may be either $+$ or $-$ or vacancy, if sign that may be called where sign is none. We may, if we please, imagine these cubes or cells to be filled with positive, negative or neutral electricity.

According to the chorographical law, (foot-note, p. 251, this Journal, Vol. 2), it ought to and would be found that the occupied portions of this parallelopiped would separate into a certain number of distinct blocks of positive and negative signs. Let us limit our attention to the first of these blocks.† The tamisage, according to the principle laid down in the remarks

* The vacant lines and columns suppressed in the rectangular tables referred to, are supposed to be supplied.

† Planes passing through that angle of the parallelopiped at which is situated the absolute constant, may be termed the planes of reference.

In order to determine whether or not a given space or cell (as we may term it) belongs to the first block, the following is the rule to be observed: 1st. If its sign is negative, it is to be rejected. 2d. If three lines be

at the end of the preceding paper, may be limited to this block, although, as a matter of fact (and for greater assurance) in deducing the tables of groundforms, it was actually applied to all the positive terms in the 11 rectangles.

An inspection of the rectangle affected with x^7 and x^8, p. 302, will show that they may be omitted as forming no part of the first positive block. In the rectangle affected with x^9, it will be found that the only terms subject to examination, *i. e.* the only terms with positive coefficients which are not *preceded* vertically or horizontally by terms with negative coefficients, are

$$\begin{array}{ccc} 2c^5d^4x^9 & 2c^5d^5x^9 & \\ 2c^7d^4x^9 & 3c^7d^5x^9 & 2c^7d^6x^9 \\ & c^9d^5x^9 & 2c^9d^6x^9. \end{array}$$

Calling any one of these terms $kc^\lambda d^\mu x^9$, it will be found, on examining the preceding rectangles, that $c^\lambda d^\mu$ will be found in one or more of them affected with a negative numerical coefficient. Consequently, these terms do not

drawn through its centre parallel to the edges of the parallelopiped *towards* the planes of reference, and any of these passes through a negative cell, it is to be rejected. 3d. In every other case, the cell (or term which occupies it) forms a part of the primary block. So to obtain the second block required for determining the syzygants of the first species, (and notice that under a general point of view groundforms may be regarded as syzygants of species zero or on the other hand and preferably syzygants of the i^{th} may be regarded as groundforms of the $(i+1)^{th}$ species) we may take any negative cell such that the three lines drawn through it parallel to the edges and towards the plane of reference shall not pass through any positive one. The *ensemble* of such constitute the second block. Then for the third block we may take the *ensemble* of positive cells not included in the first block and such that the lines through any one of them drawn as before shall not pass through a negative cell, and so on until all the cells are distributed into their respective blocks.

It may not be out of place to observe here that groundforms and syzygants may be regarded as existences and privations of existence, and the Fundamental Postulate so often previously quoted (on which the legitimacy of tamisage depends) is analogous to the assertion that free electricities of the two kinds cannot coëxist at the same time at the same point of a body. Are there not some phenomena in electricity (certain visible effects at the poles of an electrical machine or at the extremities of the electric arc) which seem to indicate that the two electricities, although mutually quelling, are not absolutely antithetical in the sense that they might be reversed throughout an environment without any change of effect of any kind resulting? Unless this is true the analogy of the relation of Groundforms and Syzygants to Positive and Negative Electricity halts on one foot. But if it be true we may perhaps see foreshadowed in the constitution of the generating function, the possibility of physical research hereafter bringing to light residual phenomena in which freer and rarer kinds of positive and negative electricity in succession will make their appearance.

Their supposed possible prototypes as yet, play no part in any developed algebraical theory, and indeed the consciousness of only a few algebraists is as yet fully awakened to a sense of their existence. If to any one the idea of physical being foreshadowed in algebraical laws should appear extravagant and visionary, let him reflect on the certain fact that the conception of chemical units as molecules composed of atoms and of the new theory of atomicity or *valence* in each essential particular might have been safely inferred as a possible hypothesis, from the ascertained laws of the constitution and mutual actions upon one another of invariantive forms. If we only allow that the so-called laws of nature have their origin in reason and are not merely arbitrary or *fiat* laws, we can very well understand how an unfailing parallelism should exist between the phenomena of the outer world and those phenomena of the pure intelligence with which algebraical science is concerned.

belong to the primary block, and, in like manner, it will be found that the rectangles subsequent to x^9 form no part of it.

The tamisage may therefore be confined to the rectangles belonging to x^0, x^1, x^2, x^3, x^4, x^5, x^6 and the only terms to be retained will be seen to be those exhibited in the following table:

	c^4d^2	$2c^4d^3$	$2c^4d^4$	c^4d^5		
	c^6d^2	$3c^6d^3$	$2c^6d^4$	c^6d^5		
	cdx	cd^2x				
	$2c^3dx$	$3c^3d^2x$	$2c^3d^3x$	c^3d^4x		
	c^5dx	$2c^5d^2x$	$3c^5d^3x$	$2c^5d^4x$	c^5d^5x	c^5d^6x
	$2c^2dx^2$	$3c^2d^2x^2$	$2c^2d^3x^2$	$c^2d^4x^2$		
	$2c^4dx^2$	$4c^4d^2x^2$	$5c^4d^3x^2$	$3c^4d^4x^2$	$c^5d^5x^2$	
	cdx^3	cd^2x^3	cd^3x^3			
c^3x^3	c^3dx^3	$3c^2d^2x^3$	$5c^3d^3x^3$	$3c^3d^4x^3$	$c^3d^5x^3$	
	c^2dx^4	$2c^2d^2x^4$	$3c^2d^3x^4$	$c^2d^4x^4$		
	cdx^5	cd^2x^5				
	d^3x^6					

Thus, it is evident at a glance that the highest order in the variables, the highest degrees in the cubic and quartic coefficients respectively, of any groundform, are 6, 4 and 5 respectively. Prior to all tamisage, 6, 4, 5 are seen to be superior limits to such order and degrees, because no powers of x, d, c figure among the above terms higher than 6, 4, 5, and a slight examination shows that some terms, containing x^6, d^4, c^5, survive the operation of the tamisage.

The number of types submitted to tamisage, it will be seen, is 45, as previously stated.

The number of forms contained under these types is 83.

The number of types absolutely abolished by the operation is 10, bringing down the number to 35; and the reduction in the total number of forms is 33, bringing down the number to 50.*

These remarks have reference *solely to the groundforms* represented by the *numerator* of the Generating Function. The denominator yields 11 ground-

* There is every reason to believe that a calculating machine might be constructed without difficulty for performing mechanically the process of *tamisage* whether simple (involving only a single variable) as for invariants of single forms or compound (involving several variables) as for covariants or invariants of systems.

forms, thus raising the total number to 61, which is the right number when the absolute constant is not counted *in* as the representative of an invariant.*

Possibly, when I may be again able to secure the services of Mr. Franklin, without whose intelligent cooperation I believe it would have been impracticable for me to have calculated the tables contained in this and the preceding number of the Journal, I shall be able to extend the limit to the order of the combined quantics. At all events, the labor of forming the tables of the combinations of 1, 2, 3, 4, 5, 6 with 6, would probably not exceed the amount which has been incurred in calculating the groundforms of a single quantic of the 9th order. The references to the *Comptes Rendus* made in the foot notes are to Vol. 88, 1^ier^ *semestre* for 1877, p. 1285, for the disproof of the existence of the *two* forms given in the accepted tables belonging to a system of two binary quartics; to Vol. 87, 2^me^ *semestre* for 1878, p. 445, and again p. 471, for the disproof of the existence of the *three* accepted superfluous forms for a system of a binary cubic and quartic, and to Vol. 89, 2^me^ *semestre* for 1879, p. 828, for the disproof of the existence of the *two* superfluous accepted forms belonging to the system of two binary cubics. The proof of the Fundamental Theorem is given as a Postscriptum in a paper in Borchardt's Journal "Sur les actions mutuelles des formes invariantives," 1878, and in a paper entitled on a Proof of the hitherto undemonstrated fundamental theorem for Invariants, in the Philosophical Magazine for the same year, 1878.

The term *Reduced Generating Function* being apt to lead to the erroneous impression that it is obtained by reducing the representative one, whereas the representative is in fact obtained from the reduced G. F. by multiplication of its numerator and denominator by a common factor, it may be well to explain that I use the appellation *reduced* with reference to the *crude form of the generating function*, the former representing that branch, or the totality of those branches, in the development of the crude form which contain no negative powers of x.

*It should be noticed that some of the entries in the Table of Groundforms, p. 303, are made up partly from the numerator and partly from the denominator as *ex. gr.* the number *3* in the column headed 3 and in the line marked *4* for the order *0*, is made up partly of the *2* in the surviving term $2d^3c^4$ of the numerator and partly of a unit taken from the term $1 - d^3c^4$ of the numerator. It is an erroneous and misleading expression into which invariantists (myself included) have fallen of speaking of a definite number, say ν, of groundforms of a certain type. The true idea is that of a unique form of that type with ν parameters. It is, so to say, a single *form* of the ν^{th} degree of plasticity or deformability or of ν dimensions in the sense in which we speak of the dimensions of space. I mean that an elastic string, an india-rubber disk and an india-rubber ball may be regarded as symbols of a groundform with one, two or three parameters respectively.

I add a few words respecting differentiants which are simply such symmetrical functions of the roots as are complete functions of the differences of the roots of the form or system of forms to which the several tables refer.

In the G. F. for differentiants for a single quantic, the coefficient of a^j represents the total number of linearly independent differentiants of the degree j belonging to a quantic of the order i; *i. e.* the total number of covariants of the degree j in the coefficients and of *all* orders in the variables, belonging to that quantic. The G. F. for differentiants can therefore be obtained from the G. F. for covariants (although not in its simplest form) by putting $x = 1$ in the latter. In like manner, for a system of quantics, the G. F. for differentiants (or to speak more precisely, its algebraical equivalent) can be obtained from the G. F. for covariants by putting $x = 1$.

To obtain the G. F. for differentiants for a single form without previously having the G. F. for covariants, we may make use of the fact that the sum of the quantities

$$(w : i, j) - (w - 1 : i, j)\ *$$

for *all* admissible values of w is equal to the value of $(w : i, j)$ for the *highest* admissible value of w. Now the *order* corresponding to the highest weight is 0 or 1†; hence the number of differentiants of the degree j belonging to a quantic of the order i is the coefficient a^jx^0 or of a^jx^1 (according as ij is even or odd) in the development of

$$\frac{1}{(1 - ax^i)(1 - ax^{i-2}) \ldots (1 - ax^{-i+2})(1 - ax^{-i})}.$$

The generating function for differentiants is therefore the sum of the multipliers of x^0 and x^1 in the development of the above fraction. (When the quantic is of even order, x^1 does not appear in the development, and the G. F. for differentiants is simply the part independent of x in the development.)

In like manner, for a system of two quantics, the G. F. for differentiants is the sum of the multipliers of x^0 and x^1 in the development of

$$\frac{1}{(1 - ax^i)(1 - ax^{i-2}) \ldots (1 - ax^{-i})(1 - ax^{i'})(1 - ax^{i'-2}) \ldots (1 - ax^{-i'})}.$$

---

* w is the weight of any covariant, j its degree in the coefficients and i the order of the quantic in the variables; and $(w : i, j)$ denotes the number of modes of composing w with j of the elements 0, 1, 2, 3, . . . i or *vice versâ* with i of the elements 0, 1, 2, 3, . . . j each any number of times repeated.

† If e is the order of the covariant in the variables $2w = ij - e$.

And we may proceed in an analogous manner when a system of forms is in question. I need hardly add that a differentiant in respect to either variable, say x, is only another name for any rational integral function of the coefficients of a quantic which, when the coefficient of the highest power of the selected variable (x) in the quantic is made equal to unity, becomes a function of the differences of its $\frac{x}{y}$ roots. Gordan's and Jordan's results concerning symbolical determinants are correlative and coëxtensive with theorems concerning root-differences, so that the method of differentiants when fully developed would lead to the substitution of actual differences or determinants for symbolical determinants in the Gordan theory, it being borne in mind that to determine the ground-covariants of a quantic or quantic system is the same question as that of determining its ground-differentiants, inasmuch as to every covariant corresponds a single differentiant, and *vice versâ.*

ERRATA

Relating to the paper entitled "*Tables of the Generating Functions and Groundforms for the Binary Quantics of the First Ten Orders*" in the preceding number of this Journal, pp. 223–251.

Page 249, line 3 from foot, *for* all those that follow *read* all those that are to follow in the next number of the Journal.

Page 250, line 13 from foot, *for* multiple of and *read* multiple of 10 and.

If no one else will undertake the task, I propose, at no distant date, to write out the scheme of operations which will furnish the system of 69 groundforms of the Binary Quantics of the 8th Order (one of the 70 stated at p. 147 being the absolute constant) and to verify the completeness of the system by the application of the Gordan test.

On the Ghosts in Rutherfurd's Diffraction-Spectra.

BY C. S. PEIRCE.

[Published by the authority of the Superintendent of the United States Coast and Geodetic Survey.]

LET there be a periodical irregularity in the ruling of a diffraction plate, so that the side of the r^{th} slit nearest a fixed line of reference parallel to the ruling shall be distant from that line by

$$\left(r-\frac{1}{2}\alpha\right)\mathrm{w}+\mathrm{e}\sin\left(r\theta-\frac{1}{2}\theta\right)$$

while the side of the same opening furthest from the line of reference is distant from it by

$$\left(r+\frac{1}{2}\alpha\right)\mathrm{w}+\mathrm{e}\sin\left(r\theta+\frac{1}{2}\theta\right).$$

This is supposing the opaque lines to have a constant breadth, $(1-\alpha)\,\mathrm{w}$.

Suppose the collimator and telescope of the spectrometer to be focused for parallel rays, and neglect the angular aperture of the slit. Let the angle of incidence be i, and the angle of emergence j. Write

$$\nu=\sin i-\sin j.$$

Then the ray which strikes the gitter at a distance x from the line of reference is longer than that which passes through the line of reference by νx. Consequently, the resultant oscillation from the r^{th} slit will be

$$\int_{(r-\frac{1}{2}\alpha)\mathrm{w}+\mathrm{e}\sin(r\theta-\frac{1}{2}\theta)}^{(r+\frac{1}{2}\alpha)\mathrm{w}+\mathrm{e}\sin(r\theta+\frac{1}{2}\theta)} \mathrm{d}x\,.\sin 2\,\frac{Vt-\nu x}{\lambda}\,\pi$$

where t is the time, V the velocity of light, and λ the wave-length. (In this paper π will be written for the ratio of the circumference to the diameter, e for the natural base, and ι for the imaginary unit.) If then we sum this for all integral values of r, we obtain an expression for the resultant oscillation from the whole gitter.

Performing the integration relatively to x, indicating the summation relative to r, and using the abbreviations

$$\omega=2\,\frac{\mathrm{w}\nu}{\lambda}\,\pi,\qquad \varepsilon\omega=2\,\frac{\mathrm{e}\nu}{\lambda}\,\pi,\qquad \tau=2\,\frac{Vt}{\lambda}\,\pi,$$

we obtain the following expression for the resultant oscillation from the whole grating:

$$\frac{\mathrm{w}}{\omega}\Sigma_r\left\{\cos\left[\varepsilon\omega\sin\left(r\theta+\frac{1}{2}\theta\right)\right].\cos\left(\tau-\frac{1}{2}\alpha\omega-r\omega\right)\right.$$
$$+\sin\left[\varepsilon\omega\sin\left(r\theta+\frac{1}{2}\theta\right)\right].\sin\left(\tau-\frac{1}{2}\alpha\omega-r\omega\right)$$
$$-\cos\left[\varepsilon\omega\sin\left(r\theta-\frac{1}{2}\theta\right)\right].\cos\left(\tau+\frac{1}{2}\alpha\omega-r\omega\right)$$
$$\left.-\sin\left[\varepsilon\omega\sin\left(r\theta-\frac{1}{2}\theta\right)\right].\sin\left(\tau+\frac{1}{2}\alpha\omega-r\omega\right)\right\}.$$

We now need a formula for developing sines and cosines of sines. For this purpose take $y=e^{\iota x}$. Then we have

$$\cos(\alpha\sin x)+\sin(\alpha\sin x).\iota=e^{\iota.\alpha\sin x}=e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}.$$

By the usual development of an exponential function, this is

$$e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}=\overset{\infty}{\underset{0}{\Sigma}}{}_p\frac{\alpha^p}{p!\,2^p}\left(y-\frac{1}{y}\right)^p,$$

and by the binomial theorem, this is,

$$e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}=\overset{\infty}{\underset{0}{\Sigma}}{}_p\frac{\alpha^p}{p!\,2^p}\overset{p}{\underset{0}{\Sigma}}{}_q(-1)^q\frac{p!}{q!\,(p-q)!}y^{p-2q}.$$

The pq^{th} term is

$$(-1)^q\frac{\alpha^p y^{p-2q}}{2^p q!\,(p-q)!}.$$

Put $m=p-2q$ and this becomes

$$(-1)^q\frac{\alpha^m y^m}{2^m}\cdot\frac{\alpha^{2q}}{4^q q!\,(m+q)!}.$$

In regard to the limits of the summation, q may have any value from zero to positive infinity, and, for every value of q, p may have any value from q to positive infinity; hence, m may have any value from $-q$ to positive infinity, and we have

$$\cos(\alpha\sin x)+\sin(\alpha\sin x).\iota=\overset{\infty}{\underset{0}{\Sigma}}{}_q(-1)^q\frac{\alpha^{2q}}{4^q.q!}\overset{\infty}{\underset{-q}{\Sigma}}{}_m\frac{\alpha^m}{2^m.(m+q)!}(\cos mx+\sin mx.\iota).$$

If m has a positive value, q may have any positive value; but if m has a negative value, q can only have any positive value greater than $-m$. Hence, we may take the terms for which m is not zero in pairs, embracing in each pair a term for which m has a positive value, M, and q has a value, Q, and also a term for which $m=-M$ and $q=M+Q$. The sum of two terms composing the pair is, then,

$$(-1)^Q\frac{\alpha^M(\cos Mx+\sin Mx.\iota)}{2^M}\cdot\frac{\alpha^{2Q}}{4^Q Q!\,(M+Q)!}$$
$$+(-1)^{M+Q}\frac{\alpha^{-M}(\cos Mx-\sin Mx.\iota)}{2^{-M}}\cdot\frac{\alpha^{2M+2Q}}{4^{M+Q}(M+Q)!\,Q!}.$$

If M is even, the value of this is

$$(-1)^Q \frac{a^M}{2^{M-1}} \frac{a^{2Q}}{4^Q Q!\,(M+Q)!} \cos Mx\,;$$

and if M is odd, its value is

$$(-1)^Q \frac{a^M}{2^{M-1}} \frac{a^{2Q}}{4^Q Q!\,(M+Q)!} \sin Mx\,.\,\iota.$$

We have then

$$\cos(\alpha \sin x) + \sin(\alpha \sin x)\,.\,\iota = \sum_0^\infty{}_q (-1)^q \frac{a^{2q}}{4^q (q\,!)^2} + \sum_1^\infty{}_m \frac{A_m a^m}{m\,!\,2^{m-1}} (\cos x + \sin x\,.\,\iota)^m;$$

where

$$A_m = \sum_0^\infty{}_q (-1)^q \frac{m\,!}{4^q q\,!\,(m+q)!} a^{2q}.$$

Performing the numerical calculations, we have

$$\begin{aligned}
\cos(\alpha \sin x) = {} & \left(1 - \frac{1}{4}\alpha^2 + \frac{1}{64}\alpha^4 - \frac{1}{2304}\alpha^6 + \frac{1}{147456}\alpha^8 - \frac{1}{14745600}\alpha^{10} + \text{etc.}\right) \\
& + \frac{1}{4}\alpha^2\left(1 - \frac{1}{12}\alpha^2 + \frac{1}{384}\alpha^4 - \frac{1}{23040}\alpha^6 + \frac{1}{2211840}\alpha^8 - \text{etc.}\right)\cos 2x \\
& + \frac{1}{192}\alpha^4\left(1 - \frac{1}{20}\alpha^2 + \frac{1}{960}\alpha^4 - \frac{1}{80640}\alpha^6 + \text{etc}\right)\cos 4x \\
& + \frac{1}{23040}\alpha^6\left(1 - \frac{1}{28}\alpha^2 + \frac{1}{1792}\alpha^4 - \text{etc.}\right)\cos 6x \\
& + \frac{1}{5160960}\alpha^8\left(1 - \frac{1}{36}\alpha^2 + \text{etc.}\right)\cos 8x \\
& + \frac{1}{1857945600}\alpha^{10}(1 - \text{etc.})\cos 10x \\
& + \text{etc.}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha \sin x) = {} & \alpha\left(1 - \frac{1}{8}\alpha^2 + \frac{1}{192}\alpha^4 \frac{1}{9216}\alpha^6 + \frac{1}{737280}\alpha^8 - \frac{1}{88473600}\alpha^{10} + \text{etc.}\right)\sin x \\
& + \frac{1}{24}\alpha^3\left(1 - \frac{1}{16}\alpha^2 + \frac{1}{640}\alpha^4 - \frac{1}{46080}\alpha^6 + \frac{1}{5160960}\alpha^8 - \text{etc.}\right)\sin 3x \\
& + \frac{1}{1920}\alpha^5\left(1 - \frac{1}{24}\alpha^2 + \frac{1}{1344}\alpha^4 - \frac{1}{129024}\alpha^6 + \text{etc.}\right)\sin 5x \\
& + \frac{1}{322560}\alpha^7\left(1 - \frac{1}{32}\alpha^2 + \frac{1}{2304}\alpha^4 - \text{etc.}\right)\sin 7x \\
& + \frac{1}{92897280}\alpha^9\left(1 - \frac{1}{40}\alpha^2 + \text{etc.}\right)\sin 9x \\
& + \frac{1}{40874803200}\alpha^{11}(1 - \text{etc.})\sin 11x \\
& + \text{etc.}
\end{aligned}$$

Making use of these series, the expression for the resultant oscillation from the gitter becomes

$$-\mathrm{w}\sum_{0\,(\text{even } m)}^{\infty} A_m \frac{\epsilon^m \omega^{m-1}}{m!\,2^{m-2}}\,\Sigma_r\Big(\cos mr\theta\,.\,\sin(r\omega-\tau)\,.\,\cos\frac{1}{2}m\theta\,.\,\sin\frac{1}{2}a\omega$$
$$+\sin mr\theta\,.\,\cos(r\omega-\tau)\,.\,\sin\frac{1}{2}m\theta\,.\,\cos\frac{1}{2}a\omega\Big)$$
$$-\mathrm{w}\sum_{1\,(\text{odd } m)}^{\infty} A_m \frac{\epsilon^m \omega^{m-1}}{m!\,2^{m-2}}\,\Sigma_r\Big(\cos mr\theta\,.\,\sin(r\omega-\tau)\,.\,\sin\frac{1}{2}m\theta\,.\,\cos\frac{1}{2}a\omega$$
$$+\sin mr\theta\,.\,\cos(r\omega-\tau)\,.\,\cos\frac{1}{2}m\theta\,.\,\sin\frac{1}{2}a\omega\Big).$$

The summation relatively to r may be effected by means of the formula,

$$\Sigma_x \sin(hx+a)\,.\,\sin(kx+b)=$$
$$\frac{-\sin\left(hx+a-\frac{1}{2}h\right).\cos\left(kx+b-\frac{1}{2}k\right).\cos\frac{1}{2}h.\sin\frac{1}{2}k+\cos\left(hx+a-\frac{1}{2}h\right).\sin\left(kx+b-\frac{1}{2}k\right).\sin\frac{1}{2}h.\cos\frac{1}{2}k.}{\cos h-\cos k}$$

For a modern gitter, it would be quite as satisfactory to consider r as infinite, and to use, in place of the above, an infinitesimal formula, which will be found in Hirsch's Integral Tables. Applying, however, the formula of finite integration, we have, as an integrated expression for the resultant oscillation from the whole gitter,

$$\frac{\mathrm{w}}{\omega}\,\frac{A_0}{1-\cos\omega}\cos\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos\frac{1}{2}(\omega-a\omega)-\cos\frac{1}{2}(\omega+a\omega)\right]$$
$$+\mathrm{w}\sum_{2\,(\textit{even } m)}^{\infty} A_m\frac{\dfrac{\epsilon^m\omega^{m-1}}{m!\,2^{m-1}}}{\cos m\theta-\cos\omega}\Big\{-\sin m\left(r\theta-\frac{1}{2}\theta\right).\sin\left(r\omega-\tau-\frac{1}{2}\omega\right).\sin m\theta.\sin\frac{1}{2}(\omega-a\omega)$$
$$+\cos m\left(r\theta-\frac{1}{2}\theta\right).\cos\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos m\theta\,.\,\cos\frac{1}{2}(\omega-a\omega)-\cos\frac{1}{2}(\omega+a\omega)\right]\Big\}$$
$$+\mathrm{w}\sum_{1\,(\textit{odd } m)}^{\infty} A_m\frac{\dfrac{\epsilon^m\omega^{m-1}}{m!\,2^{m-1}}}{\cos m\theta-\cos\omega}\Big\{\cos m\left(r\theta-\frac{1}{2}\theta\right).\cos\left(r\omega-\tau-\frac{1}{2}\omega\right).\sin m\theta.\sin\frac{1}{2}(\omega-a\omega)$$
$$-\sin m\left(r\theta-\frac{1}{2}\theta\right).\sin\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos m\theta\,.\,\cos\frac{1}{2}(\omega-a\omega)-\cos\frac{1}{2}(\omega+a\omega)\right]\Big\}.$$

This expression may be simplified by writing

$$x=\frac{1}{2}(\omega+m\theta)\,,$$
$$y=\frac{1}{2}(\omega-m\theta)\,;$$

so that

$$\sin\left[\left(r-\frac{1}{2}\right)m\theta\right].\sin\left[\left(r-\frac{1}{2}\right)\omega-\tau\right]=\frac{1}{2}\cos[(2r-1)y-\tau]-\frac{1}{2}\cos[(2r-1)x-\tau]$$

$$\cos\left[\left(r-\frac{1}{2}\right)m\theta\right].\cos\left[\left(r-\frac{1}{2}\right)\omega-\tau\right]=\frac{1}{2}\cos[(2r-1)y-\tau]+\frac{1}{2}\cos[(2r-1)x-\tau].$$

We have also to observe that

$$\mp\sin m\theta.\sin\frac{1}{2}(\omega-\alpha\omega)+\cos m\theta.\cos\frac{1}{2}(\omega-\alpha\omega)-\cos\frac{1}{2}(\omega+\alpha\omega)$$

$$=\cos\left[\frac{1}{2}(\omega-\alpha\omega)\pm m\theta\right]-\cos\frac{1}{2}(\omega+\alpha\omega)=+2\sin\frac{1}{2}(\omega\pm m\theta)\sin\frac{1}{2}(\alpha\omega\mp m\theta).$$

Thus, the quantity in parenthesis, under the sum for even values of m, reduces to

$$\cos[(2r-1)y-\tau].\sin\frac{1}{2}(\omega+m\theta).\sin\frac{1}{2}(\alpha\omega-m\theta)$$

$$+\cos[(2r-1)x-\tau].\sin\frac{1}{2}(\omega-m\theta).\sin\frac{1}{2}(\alpha\omega+m\theta),$$

and the corresponding quantity for odd values of m, to

$$-\cos[(2r-1)y-\tau].\sin\frac{1}{2}(\omega+m\theta).\sin\frac{1}{2}(\alpha\omega-m\theta)$$

$$+\cos[(2r-1)x-\tau].\sin\frac{1}{2}(\omega-m\theta).\sin\frac{1}{2}(\alpha\omega+m\theta).$$

The integral is to be taken between limiting values of r, say r_1 and r_2. Let the whole number of openings in the gitter be R, so that

$$R=r_2-r_1.$$

Then, a second equation to determine r_1 and r_2 may be assumed arbitrarily without affecting the result. Let this equation be

$$r_2+r_1=1.$$

Then

$$(2r_2-1)=-(2r_1-1)=R.$$

Now r occurs only in the factors

$$\cos[(2r-1)y-\tau]=\cos(2r-1)y.\cos\tau+\sin(2r-1)y.\sin\tau$$

and

$$\cos[(2r-1)x-\tau]=\cos(2r-1)x.\cos\tau+\sin(2r-1)x.\sin\tau.$$

Taken between these limits, these factors will be respectively,

$$2\sin Ry.\sin\tau,$$

$$2\sin Rx.\sin\tau.$$

Applying these reductions, and also remembering that

$$\cos m\theta-\cos\omega=2\sin x\sin y,$$

the expression for the resultant oscillation from the whole gitter reduces to

$$\sin\tau\,.\,\mathrm{w}\sum_{-\infty}^{+\infty}{}_m A_m \frac{\epsilon^m\omega^{m-1}}{m!\,2^{m-1}} \frac{\sin\frac{1}{2}R(\omega+m\theta)}{\sin\frac{1}{2}(\omega+m\theta)} \sin\frac{1}{2}(a\omega+m\theta)\,,$$

where, in summing for negative values of m, positive values are to be taken in the coefficients, and where terms arising from odd negative values of m in the parenthesis are to have the opposite sign, and where the term in $m=0$ is to have only half the above value.

We have now to study the principal maxima of the amplitude of this oscillation, for varying ω. Taking each term of the series separately, we observe that one factor of it, namely,

$$\frac{\sin\frac{1}{2}R(\omega+m\theta)}{\sin\frac{1}{2}(\omega+m\theta)},$$

reaches a maximum when

$$\omega+m\theta=2N\pi,$$

and this maximum value is R. Now R is a number amounting to several thousand, while a is less than unity. Hence, the maximum of the whole term will be very nearly at the same place, and one of the maxima of the sum of all the terms will also be nearly in that place.

To ascertain the precise position of the maximum of any one term, put

$$\omega=2N\pi-m\theta+\delta\omega\,.$$

Then, neglecting the cube of $\delta\omega$, in comparison with unity, we have

$$\sin\frac{1}{2}R(\omega+m\theta)=\pm\sin\frac{1}{2}R\delta\omega=\pm\frac{1}{2}R\delta\omega\mp\frac{1}{48}R^3(\delta\omega)^3$$

$$\sin\frac{1}{2}(\omega+m\theta)=\pm\sin\frac{1}{2}\delta\omega=\pm\frac{1}{2}\delta\omega\mp\frac{1}{48}(\delta\omega)^3$$

$$\frac{\sin\frac{1}{2}R(\omega+m\theta)}{\sin\frac{1}{2}(\omega+m\theta)}=\pm\frac{\sin\frac{1}{2}R\delta\omega}{\sin\frac{1}{2}\delta\omega}=\pm R\mp\frac{1}{24}(R^3-R)(\delta\omega)^2.$$

As for $\sin\frac{1}{2}(a\omega+(-1)^m m\theta)$, it may have any value whatever from -1 to $+1$, according to the magnitude of a. But it is when it vanishes that the maximum is at the greatest value of $\delta\omega$. Let us then suppose

$$\sin\frac{1}{2}(a\omega+(-1)^m m\theta)=\pm\frac{1}{2}a\delta\omega\mp\frac{1}{48}a^3(\delta\omega)^3.$$

Finally, there is the factor ω^{m-1}. Dividing this by $(2N\pi - m\theta)^{m-1}$, we have

$$\left(\frac{\omega}{2N\pi - m\theta}\right)^{m-1} = 1 + (m-1)(2N\pi - m\theta)^{-1}\delta\omega + \frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2}(\delta\omega)^2;$$

finally, multiplying together the quantities thus obtained, we find as that factor of the mth term which contains $(\delta\omega)$

$$\delta\omega + (m-1)(2N\pi - m\theta)^{-1}\delta\omega^2 + \left\{\frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2} - \frac{1}{24}\alpha^2 - \frac{1}{24}(R^2-1)\right\}(\delta\omega)^3.$$

Differentiating, we find as the equation for determining the value of $\delta\omega$ at the maximum of the mth term

$$1 + 2(m-1)(2N\pi - m\theta)^{-1}\delta\omega + 3\left\{\frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2} - \frac{1}{24}\alpha^2 - \frac{1}{24}(R^2-1)\right\}(\delta\omega)^2 = 0.$$

If we neglect $\frac{1}{R^2}$, the solution of this equation is

$$\delta\omega = \frac{8(m-1)}{R^2(2N\pi - m\theta)}.$$

It will be seen that $\delta\omega$ is zero when $m = 1$, and that for the principal spectrum, for which $m = 0$, if $R = 1000$, $\frac{\delta\omega}{\omega}$ is altogether inappreciable, but if $R = 100$, $\frac{\delta\omega}{\omega} =$ about $\frac{1}{50000}$ for the first order, which displaces the spectrum by about $\frac{1}{50}$ part of the distance between the two D lines.

We have now to consider how far the maxima of the sum of the series representing the oscillation may differ from those of the single terms. A term will have the most influence in displacing a maximum when it is itself nearly zero, or more accurately when its differential coefficient relatively to ω is at a maximum. As ω increases by 2π so as to pass from one principal maximum of oscillation to another, $R\omega$ passes R times through 2π, so that the term passes through as many maxima and minima. Then the differential coefficient relative to ω of the sum of all the terms will be the greatest for a value of ω such that

$$\omega + m_0\theta = 2N\pi,$$

(m_0 being a given value of m), when, in addition to the above equation, we have

$$R\theta = 4N\pi.$$

In this case, the differential coefficient of the mth term of the expression for the oscillation will be

$$\frac{R}{\omega} m! \left(\frac{\epsilon\omega}{2}\right)^2 \frac{1}{\sin\frac{1}{2}(\omega + m\theta)}.$$

It will be sufficiently accurate to put

$$\sin\frac{1}{2}(\omega + m\theta) = \frac{1}{2}(m - m_0)\,\theta.$$

Then it is plain that, were the term for $m = 0$ of the same value as the others, the total differential coefficient would be

$$\frac{R}{\omega} m_0\, e^{\left(\frac{\epsilon\omega}{2}\right)}$$

Owing, however, to the term for $m = 0$ having only half the value given by the formula, the value is

$$\frac{R}{\omega} m_0 \left(e^{\left(\frac{\epsilon\omega}{2}\right)} - \frac{1}{2}\right).$$

In consequence of the differential coefficient having this value, the maximum will not occur exactly at the value of α for which

$$\omega + m_0\theta = 2N\pi,$$

but will be shifted along to the point where the differential coefficient of the m_0th term is equal to the negative of the differential coefficient just found. If $\delta\omega$ is the amount of the shifting, the m_0th term of the oscillation (R being very large) is

$$\frac{\sin.\frac{R}{2}\,\delta\omega}{\delta\omega}.$$

The differential coefficient of this is

$$\frac{1}{4}\,\frac{\sin.R.\delta\omega - R\delta\omega}{(\delta\omega)^2},$$

and the equation to determine $\delta\omega$ is

$$\frac{1}{4}\,\frac{\sin.R\delta\omega - R\delta\omega}{(\delta\omega)^2} = \frac{R}{\omega} m_0 \left(1 - e^{\frac{\epsilon\omega}{2}}\right).$$

In the worst case, this becomes

$$\delta\omega = \frac{24}{R^2} m_0 \left(e^{\frac{\epsilon\omega}{2}} - 1\right).$$

It thus appears that the position of the principal spectrum will not be disturbed by the circumstance here considered, and that the distance between the successive ghosts will be very slightly altered.

It is to be remarked that, when two spectral lines fall very near together, they will be attracted to one another in consequence of the mixture of light

by a sensible amount. This will especially affect the position of a faint line near a very intense one.

The Phenomena.

Mr. Rutherfurd's diffraction-plates are ruled with a machine which is described by Professor A. M. Mayer in the article "Spectrum," in the second edition of *Appleton's Cyclopædia.* In consequence of the periodic error of the screw, a periodic inequality is produced in the ruling. This is shewn by putting a gitter into the spectrometer, illuminating it with homogeneous light, and observing it without the eye-piece, when it appears striped. If the eye-piece is replaced and a real solar spectrum is thrown on the slit-plate, of such purity that the light admitted into the slit varies only by a few ten-thousandths of a micron in wave-length, the maxima of light which have been investigated above appear as repetitions of the principal spectrum, in which even the fine lines due to the solar atmosphere are distinctly visible.

The positions of some of these "ghosts," or repetitions of the principal spectrum, have been carefully measured in order to test the theory.

Measures of the Positions of the Ghosts.

To determine whether the screw of the filar micrometer had the same pitch throughout its length, the distance between D_1 and D_2 was measured on different places on the screw. Gitter: speculum metal 681 lines to the millimeter. Second order, principal spectrum. Readings given are means of five pointings each. Date: 1879, July 3.

Place on the Screw	First End.		Second End.		Second End.		First End.	
Line of Spectrum	D	D_2	D_1	D_2	D_2	D_1	D_2	D_1
Micrometer reading	$7^r.109$	$7^r.947$	$12^r.108$	$12^r.943$	$12^r.937$	$12^r.102$	$7^r.925$	$7^r.089$
Distance of Lines	$0^r.838$		$0^r.835$		$0^r.835$		$0^r.836$	

The following were made with a speculum-metal gitter of 340½ teeth to the millimeter. Each reading given is the mean of five pointings. Date: 1879, July 3. To pass from one spectrum to another the gitter alone was turned.

Order of Spectrum	Order IV.						
Number of Ghost	Ghost, − 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$8^r.241$	$9^r.330$	$9^r.723$	$10^r.800$	$11^r.187$	$12^r.272$	
Distance $(D_1 - D_2)$	$1^r.089$		$1^r.077$		$1^r.085$		$1^r.084$
Distance of successive Ghosts $\{D_2$		$1^r.482$		$1^r.464$			1.473
Distance of successive Ghosts $\{D_1$		1.470		1.472			1.471
Mean		1.476		1.468			1.472

Order of Spectrum	Order V.								
Number of Ghost	Ghost, − 1.			Ghost, 0.			Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1		D_2	D_1		D_2	D_1	
Micrometer reading	$7^{r}.847$	$9^{r}.337$		$9^{r}.466$	$10^{r}.962$		$11^{r}.090$	$12^{r}.575$	
Distance ($D_1 - D_2$)	$1^{r}.490$			$1^{r}.496$			$1^{r}485$		$1^{r}.490$
Distance of successive Ghosts D_2			$1^{r}.619$			$1^{r}.624$			1 .621
Distance of successive Ghosts D_1			1 .625			1 .613			1 .619
Mean			1 .622			1 .618			1 .620

Order of Spectrum	Order VI.								
Number of Ghost	Ghost, − 1.			Ghost, 0.			Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1		D_2	D_1		D_2	D_1	
Micrometer reading	$7^{r}.378$	$9^{r}.421$		$9^{r}.265$	$11^{r}.304$		$11^{r}.152$	$13^{r}.173$	
Distance ($D_1 - D_2$)	$2^{r}.043$			$2^{r}.039$			$2^{r}.021$		$2^{r}.034$
Distance of successive Ghosts D_2			$1^{r}.887$			$1^{r}.887$			1 .887
Distance of successive Ghosts D_1			1 .883			1 .869			1 .876
Mean			1 .885			1 .878			1 .881

Order of Spectrum	Order VII.								
Number of Ghost	Ghost, − 1.			Ghost, 0.			Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1		D_2	D_1		D_2	D_1	
Micrometer reading	$6^{r}.637$	$9^{r}.595$		$8^{r}.955$	$11^{r}.876$		$11^{r}.262$	$14^{r}.191$	
Distance ($D_1 - D_2$)	$2^{r}.958$			$2^{r}.921$			$2^{r}929$		$2^{r}.936$
Distance of successive Ghosts D_2			$2^{r}.318$			$2^{r}.307$			2 .312
Distance of successive Ghosts D_1			2 .281			2 .315			2 .298
Mean			2 .299			2 .311			2 .305

Order of Spectrum	Order VIII.								
Number of Ghost	Ghost, − 1.			Ghost, 0.			Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1		D_2	D_1		D_2	D_1	
Micrometer reading	$4^{r}.737$	$9^{r}.467$		$8^{r}.002$	$12^{r}.680$		$11^{r}.256$	$15^{r}.885$	
Distance ($D_1 - D_2$)	$4^{r}.730$			$4^{r}.678$			$4^{r}.629$		$4^{r}.679$
Distance of successive Ghosts D_2			$3^{r}.265$			$3^{r}.254$			3 .261
Distance of successive Ghosts D_1			3 .213			3 .205			3 .209
Mean			3 .239			3 .229			3 .234

Order of Spectrum	Order IX.								
Number of Ghost	Ghost, − 1.			Ghost, 0.			Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1		D_2	D_1		D_2	D_1	
Micrometer reading	$\bar{6}^{r}.865$*	$9^{r}.403$		$4^{r}.281$	$16^{r}.977$		$12^{r}.075$	$24^{r}.435$	
Distance ($D_1 - D_2$)	$12^{r}.538$			$12^{r}.696$			$12^{r}.360$		$12^{r}.532$
Distance of successive Ghosts D_2			$7^{r}.416$			$7^{r}.794$			7 .605
Distance of successive Ghosts D_1			7 .574			7 .458			7 .516
Mean			7 .495			7 .626			7 .560

* Read $\bar{5}^{r}.865$. Either this is an erroneous reading, or a wrong line was measured.

The following measures were made with a metal gitter of 681 lines to the millimeter. Dates: 1879, June 20 and July 2.

Order of Spectrum					Order I.						
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$7^{r}.286$	$7^{r}.799$	$8^{r}.632$	$9^{r}.112$	$9^{r}.925$	$10^{r}.383$	$11^{r}.196$	$11^{r}.664$	$13^{r}.496$	$12^{r}.928$	
$D_1 - D_2$	$0^{r}.513$		$0^{r}.480$		$0^{r}.458$		$0^{r}.468$		$0^{r}.432$		$0^{r}.470$
Distance of successive Ghosts D_2		$1^{r}.346$		$1^{r}.293$		$1^{r}.271$		$1^{r}.300$			1 .302
Distance of successive Ghosts D_1		1 .313		1 .271		1 .281		1 .264			1 .282
Mean		1. 330		1 .282		1 .276		1 .282			1 .292
Order of Spectrum					Order II.						
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$5^{r}.312$	$2^{r}.482$	$6^{r}.907$	$8^{r}.059$	$8^{r}.477$	$9^{r}.627$	$10^{r}.067$	$11^{r}.191$	$11^{r}.632$	$12^{r}.752$	
$D_1 - D_2$	$1^{r}.170$		$1^{r}.152$		$1^{r}.150$		$1^{r}.124$		$1^{r}.120$		$1^{r}.143$
Distance of successive Ghosts D_2		$1^{r}.595$		$1^{r}.570$		$1^{r}.590$		$1^{r}.565$			1 .580
Distance of successive Ghosts D_1		1 .577		1 .568		1 .564		1 .561			1 .568
Mean		1 .586		1 .569		1 .577		1 .563			1 .574
Order of Spectrum					Order III.						
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$4^{r}.593$	$6^{r}.896$	$6^{r}.713$	$9^{r}.053$	$8^{r}.876$	$11^{r}.205$	$10^{r}.989$	$13^{r}.280$	$13^{r}.057$	$15^{r}.308$	
$D_1 - D_2$	$2^{r}.303$		$2^{r}.340$		$2^{r}.329$		$2^{r}.291$		$2^{r}.251$		$2^{r}.303$
Distance of successive Ghosts D_2		$2^{r}.120$		$2^{r}.163$		$2^{r}.113$		$2^{r}.068$			2 .116
Distance of successive Ghosts D_1		2 .157		2 .152		2 .075		2 .028			2 .103
Mean		2 .138		2 .158		2 .094		2 .048			2 .110

The following measures were made on spectra produced by a narrow silvered-glass plate of 681 lines to the millimeter. This gitter was selected as making unusually bright ghosts. The refraction by the glass must sensibly displace the ghosts. The two sodium lines, and the nickel line between them, were observed. Date: 1879, June 19.

Order of Sp.	ORDER I.									
No. of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.	Means.
Line of Sp.	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1		
Mic. reading	$9^r.076$	$9^r.590$	$10^r.405$	$10^r.886$	$11^r.695$	$12^r.164$	$12^r.976$	$13^r.439$		
$D_1 - D_2$	$0^r.513$		$0^r.480$		$0^r.469$		$0^r.463$			$0^r.481$
Dist. suc. Ghosts D_2		$1^r.329$		$1^r.290$		$1^r.281$				1 .300
Dist. suc. Ghosts D_1		1 .296		1 .279		1 .275				1 .283
Mean		1 .312		1 .284		1 .278				1 .291

Order of Sp.	ORDER II.														
No. of Ghost	Ghost, — 2.		Ghost, — 1.			Ghost, 0.			Ghost, + 1.			Ghost, + 2.			Means.
Line of Sp.	D_2	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	
Mic. reading	$5^r.451$	$6^r.531$	$6^r.916$	$7^r.465$	$7^r.981$	$8^r.370$	$8^r.926$	$9^r.439$	$9^r.838$	$10^r.379$	$10^r.897$	$11^r.289$	$11^r.837$	$12^r.345$	
$Ni - D_2$	$1^r.080$		$0^r.549$			$0^r.556$			$0^r.541$			$0^r.548$			$0^r.548$
$D_1 - Ni$				$0^r.516$			$0^r.513$			$0^r.518$			$0^r.508$		0 .514
Dist. success. Ghosts D_2		$1^r.465$			$1^r.454$			$1^r.468$			$1^r.451$				1 .460
Dist. success. Ghosts Ni					1 .461			1 .453			1 .458				1 .457
Dist. success. Ghosts D_1		1 .450			1 .458			1 .458			1 .448				1 .454
Mean		1 .458			1 .458			1 .460			1 .452				1 .457

Order of Sp.	ORDER III.															
No. of Ghost	Ghost, — 2.			Ghost, — 1.			Ghost, 0.			Ghost, + 1.			Ghost, + 2.			Means.
Line of Sp.	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	
Mic. reading	$6^r.822$	$7^r.874$	$8^r.843$	$8^r.671$	$9^r.715$	$10^r.693$	$10^r.515$	$11^r.559$	$12^r.535$	$12^r.365$	$13^r.399$	$14^r.372$	$14^r.192$	$15^r.228$	$16^r.198$	
Ni — D_2		$1^r.052$			$1^r.044$			$1^r.044$			$1^r.034$			$1^r.036$		$1^r.044$
D_1 — Ni			$0^r.969$			$0^r.978$			$0^r.976$			$0^r.973$			$0^r.970$	0 .973
Dist. success. Ghosts D_2			$1^r.849$			$1^r.844$			$1^r.850$			$1^r.827$				1 .842
Dist. success. Ghosts Ni			1 .841			1 .844			1 .840			1 .829				1 .838
Dist. success. Ghosts D_1			1 .850			1 .842			1 .837			1 .826				1 .839
Mean			1 .847			1 .843			1 .842			1 .827				1 .840

Order of Sp.	ORDER IV.										
No. of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.		Means.
Line of Sp.	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	
Mic. reading	$3^r.614$	$8^r.222$	$6^r.777$	$11^r.361$	$9^r.939$	$14^r.473$	$13^r.064$	$17^r.581$	$16^r.148$	$20^r.616$	
D_1 — D_2	$4^r.608$		$4^r.584$		$4^r.534$		$4^r.517$		$4^r.468$		$4^r.542$
Dist. success. Ghosts D_2		$3^r.163$		$3^r.162$		$3^r.125$		$3^r.084$			3 .134
Dist. success. Ghosts D_1		3 .139		3 .112		3 .108		3 .035			3 .096
Mean		3 .151		3 .137		3 .116		3 .060			3 .115

The following measures were made upon C, with the metal gitter of 681 lines per mm. The distance of the fine line $\lambda = 6567.91$ (Å.) from C was measured in the principal spectrum to determine the dispersion. Date: 1879, July 1.

Order I.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line.		C.
$8^{r}.241$		$9^{r}.792$		$11^{r}.289$	$9^{r}.255$		$9^{r}.801$
	$1^{r}.551$		$1^{r}.497$			$0^{r}.546$	

Order II.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line		C.
$8^{r}.054$		$9^{r}.941$		$11^{r}.774$	$8^{r}.629$		$9^{r}.960$
	$1^{r}.887$		$1^{r}.833$			$1^{r}.331$	

Order III.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line.	
$7^{r}.115$		$10^{r}.010$		$12^{r}.734$	$7^{r}.054$	
	$2^{r}.895$		$2^{r}.724$			$2^{r}.956$

The following measure was made upon F, with the same gitter. The mean of lines 4870.47 and 4871.29 was pointed on to determine the dispersion. Date: 1879, July 1,

Order II.

Double. Ghost, 0.		F. Ghost, 0.		Ghost, + 1.
$8^{r}.617$		$10^{r}.484$		$11^{r}.683$
	$1^{r}.867$		$1^{r}.190$	

The above measures satisfy the theory moderately well. Thus, according to theory, the product of the ratio of the distance of successive ghosts to the distance between the D line by the order of the spectrum should be constant, and should be twice as great for the gitter of 340½ lines to the millimeter as for that of 681 lines to the millimeter. Now this product is as follows:

Metal Gitter of 340½ *lines to the mm.*

Order	IV.	$5.43 = 2 \times 2.72$
"	V.	$5.44 = 2 \times 2.72$
"	VI.	$5.55 = 2 \times 2.77$
"	VII.	$5.50 = 2 \times 2.75$
"	VIII.	$5.53 = 2 \times 2.76$
"	IX.	$5.46 = 2 \times 2.73$

Metal Gitter of 681 *lines to the mm.*

Order	I.	2.75
"	II.	2.75
"	III.	2.75

Silvered-glass Gitter of 681 *lines to the mm.*

Order	I.	2.68
"	II.	2.74
"	III.	2.74
"	IV.	2.74

It is evident that the value which best satisfies the observations lies between 2.74 and 2.75. This ratio multiplied by the ratio of the difference of wave-length of the D lines to their mean wave-length, should give the number of lines of the finer gitters to a period of the inequality. This, from the construction of the ruling-machine, is known to be nearly, but not exactly, 360. Mr. Chapman, who works with the machine, has made certain observations, from which it would appear that the period differs about 1 per cent. from 360. The product of the ratios just mentioned (taking 2.746 for the first) is 357. This is therefore a happy confirmation of the theory.

Next, using the value 2.746, I calculate by least squares the best values of the distance of the D lines and the distance of consecutive ghosts in each order. In this way, we shall be able to judge whether the discrepancies of the observations from theory are, or are not, greater than their probable errors. The results are as follows:

Metal Gitter of $340\frac{1}{2}$ *lines to the mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
IV.	$1^r.084$	$1^r.076$	$+0^r.008$	$1^r.472$	$1^r.477$	$-0^r.005$
V.	1.490	1.481	+ 0.009	1.620	1.626	− 0.006
VI.	2.034	2.045	− 0.011	1.881	1.872	+ 0.009
VII.	2.936	2.936	0.000	2.305	2.305	0.000
VIII.	4.679	4.691	− 0.012	3.234	3.221	+ 0.013
IX.	12.532	12.485	+ 0.047	7.560	7.618	− 0.058

Metal Gitter of 681 *lines to the mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
I.	0r.470	0r.470	0r.000	1r.292	1r.292	0r.000
II.	1 .143	1 .147	— 0 .004	1 .574	1 .573	+ 0 .001
III.	2 .303	2 .304	— 0 .001	2 .110	2 .109	+ 0 .001

Silvered-glass Gitter of 681 *lines per mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
I.	0r.481	0r.470	+ 0r.011	1r.291	1r.292	— 0r.001
II.	1 .062	1 .063	— 0 .001	1 .457	1 .457	0 .000
III.	2 .017	2 .021	— 0 .004	1 .840	1 .838	+ 0 .002
IV.	4 .542	4 .544	— 0 .002	3 .115	3 .113	+ 0 .002

The discrepancies between observation and calculation are, in the case of the observations with the coarse-ruled plate in the 4th to the 7th orders, inclusive, pretty well accounted for by the attractions of neighboring lines. This is shown by the subjoined table. In other cases, there are large discrepancies amounting to 7″, or even more, which cannot be so accounted for, and which vastly exceed the errors of observation. Thus, it will almost invariably be found that the ghosts of D_1 are closer together than those of D_2, and that the distances decrease as m increases algebraically. The measures of the ghosts of C and F indicate a much longer period in the inequality. Some attempts have been made to measure the brilliancy of the ghosts. These only roughly agree with the theory.

DETAILED COMPARISON OF CALCULATION AND OBSERVATION.

Metal Gitter of 340½ *lines per mm.*

Order IV.

	Obs.	Calc.	O. — C.	
$G - 1, D_2$	8r.241	8r.244	—.003	
$G - 1, D_1$	9 .330	9 .320	+.010	Carried toward G 0, D_2.
$G\,0, D_2$	9 .723	9 .721	+.002	
$G\,0, D_1$	10 .800	10 .797	+.003	
$G + 1, D_2$	11 .187	11 .198	—.011	Carried toward G 0, D_1.
$G + 1, D_1$	12 .272	12 .274	—.002	

Order V.

$G-1, D_2$	$7^r.847$	$7^r.846$	$+.001$	
$G-1, D_1$	9.337	9.327	$+.010$	Carried toward $G\,0, D_2$.
$G\,0, D_2$	9.466	9.472	$-.006$	Carried toward $G-1, D_1$.
$G\,0, D_1$	10.962	10.953	$+.009$	Carried toward $G+1, D_2$.
$G+1, D_2$	11.090	11.098	$-.008$	Carried toward $G\,0, D_1$.
$G+1, D_1$	12.575	12.579	$-.004$	

Order VI.

$G-1, D_2$	$7^r.387$	$7^r.388$	$-.001$	
$G\,0, D_2$	9.265	9.260	$+.005$	Carried a little toward $G-1, D_1$.
$G-1, D_1$	9.421	9.433	$-.012$	Carried toward $G\,0, D_2$.
$G+1, D_2$	11.152	11.132	$+.020$	Carried toward $G\,0, D_1$.
$G\,0, D_1$	11.304	11.305	$-.001$	Carried a little toward $G+1, D_2$.
$G+1, D_1$	13.173	13.177	$-.004$	

Order VII.

$G-1, D_2$	$6^r.637$	$6^r.646$	$-.009$	Single pointings discordant. Rejecting worst obs. $= 6^r.643$.
$G\,0, D_2$	8.955	8.951	$+.005$	Carried toward $G-1, D_2$.
$G-1, D_1$	9.595	9.582	$+.013$	Should be carried toward $G\,0, D_2$.
$G+1, D_2$	11.262	11.256	$+.006$	Carried toward $G\,0, D_1$.
$G\,0, D_1$	11.876	11.887	$-.011$	Carried toward $G+1, D_2$.
$G+1, D_1$	14.191	14.192	$-.001$	

Order VIII.

$G-1, D_2$	$4^r.737$	$4^r.771$	$-.034$	
$G\,0, D_2$	8.002	7.992	$+.010$	
$G-1, D_1$	9.467	9.462	$+.005$	
$G+1, D_2$	11.256	11.213	$+.043$	No distinct attractions.
$G\,0, D_1$	12.680	12.683	$-.003$	
$G+1, D_1$	15.885	15.904	$-.019$	

Order IX.

$G-1, D_2$	$\bar{6}^r.865$	$\bar{6}^r.812$	$+.053$
$G\,0, D_2$	4.281	4.430	$-.149$
$G-1, D_1$	9.403	9.297	$+.106$
$G+1, D_2$	12.075	12.048	$+.027$
$G\,0, D_1$	16.977	16.915	$+.062$
$G+1, D_1$	24.435	24.533	$-.098$

Metal Gitter 681 *lines per mm.*

Order 1.

				O. — C. —.012	
G − 2, D_2	7^r.286	7^r.323	−.037	−.049	Noted at the time of obs. extremely uncertain.
G − 2, D_1	7 .799	7 .793	+.006	−.006	General attraction toward the middle.
G − 1, D_2	8 .632	8 .615	+.017	+.005	
G − 1, D_1	9 .112	9 .085	+.027	+.015	
G 0, D_2	9 .925	9 .907	+.018	+.006	
G 0, D_1	10 .383	10 .377	+.006	−.006	
G + 1, D_2	11 .196	11 .199	−.003	−.015	
G + 1, D_1	11 .664	11 .669	−.005	−.017	
G + 2, D_2	12 .496	12 .491	+.005	−.007	
G + 2, D_1	12 .928	12 .961	−.033	−.045	

Order II.

				O. — C. —.004
G − 2, D_2	5^r.312	5^r.331	−.019	−.023
G − 2, D_1	6 .482	6 .478	+.004	−.000
G − 1, D_2	6 .907	6 .904	+.003	−.001
G − 1, D_1	8 .059	8 .051	+.008	+.004
G 0, D_2	8 .477	8 .477	.000	−.004
G 0, D_1	9 .627	9 .624	+.003	−.001
G + 1, D_2	10 .067	10 .050	+.017	+.013
G + 1, D_1	11 .191	11 .197	−.006	−.010
G + 2, D_2	11 .632	11 .623	+.009	+.005
G + 2, D_1	12 .752	12 .770	−.018	−.022

Order III.

G − 2, D_2	4^r.593	4^r.627	−.034
G − 1, D_2	6 .713	6 .736	−.023
G − 2, D_1	6 .896	6 .931	−.035
G 0, D_2	8 .876	8 .845	+.031
G − 1, D_1	9 .053	9 .040	+.013
G + 1, D_2	10 .989	10 .954	+.035
G 0, D_1	11 .205	11 .149	+.056
G + 2, D_2	13 .057	13 .063	−.006
G + 1, D_1	13 .280	13 .258	+.022
G + 2, D_1	15 .308	15 .367	−.059

On a Theorem for Expanding Functions of Functions.

BY EMORY MCCLINTOCK, *Milwaukee, Wisconsin.*

To expand any function of fx, say ϕfx, where

$$fx = a + bx + \frac{1}{2}cx^2 + \frac{1}{2.3}dx^3 + \ldots,$$

it is customary to employ the method of derivation devised by Arbogast. By this method,

$$\phi fx = \phi a + xd\phi a + \frac{1}{2}x^2d^2\phi a + \ldots,$$

where, after the differentiations indicated are performed, the differentials da, d^2a, d^3a, ..., are to be replaced by b, c, d, ... A more direct solution of the problem will, I think, be derived, as a special case, from the following general theorem, which would seem to be new:

$$\phi\,(fx, f'x, f''x\ldots) = \left(1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \frac{1}{2.3}x^3\mathrm{D}^3 + \ldots\right)\phi\,(a, b, c\ldots). \quad (1)$$

Here

$$\mathrm{D} = b\frac{d}{da} + c\frac{d}{db} + d\frac{d}{dc} + \ldots,*$$

D^n represents n successive performances of the operation D, and

$$f'x = \frac{dfx}{dx} = b + cx + \frac{1}{2}dx^2 + \frac{1}{2.3}ex^3 + \ldots,$$

$$f''x = c + dx + \frac{1}{2}ex^2 + \frac{1}{2.3}fx^3 + \ldots,$$

and so on. To prove this theorem, which is a simple consequence of that of Maclaurin, it is only necessary to write

$$\phi\,(fx, f'x, f''x\ldots) = \left(1 + x\frac{d}{d0} + \frac{1}{2}x^2\frac{d^2}{d0^2} + \ldots\right)\phi\,(f0, f'0, f''0\ldots),$$

and to observe that, as regards any function of $f0, f'0\ldots$, say ψ,

$$\frac{d\psi}{d0} = f'0\frac{d\psi}{df0} + f''0\frac{d\psi}{df'0} + \ldots,$$

* The letter d is here used in two senses, neither of which is likely to be misunderstood. The same may be said of the letter f.

whence, since $f0$, $f'0$, &c., are respectively $a, b \ldots$, the symbol $\frac{d}{d0}$ may be replaced by D.

A more detailed demonstration may be derived from Taylor's theorem. Thus, if z represent $\phi\,(fz, f'z, f''z \ldots)$,

$$\phi\,[f\,(z+x), f'\,(z+x), \ldots] = \mathrm{z} + x\,\frac{d\mathrm{z}}{dz} + \frac{1}{2}\,x^2\,\frac{d^2\mathrm{z}}{dz^2} + \ldots. \qquad (2)$$

The operation $\frac{d}{dz}$ is to be performed successively on z, $\frac{d\mathrm{z}}{dz}$, $\frac{d^2\mathrm{z}}{dz^2} \ldots$, all functions of fz, $f'z \ldots$, and may therefore be decomposed, on the principle of partial differentiation, into

$$\frac{d}{dz} = f'z\,\frac{d}{dfz} + f''z\,\frac{d}{df'z} + \ldots.$$

Let $z = 0$, and let $\frac{d}{dz}$ be represented by D; then, since $f0 = a$, $f'0 = b \ldots$,

$$\mathrm{D} = b\,\frac{d}{da} + c\,\frac{d}{db} + \ldots,$$

and (2) becomes

$$\phi\,(fx, f'x \ldots) = (1 + x\mathrm{D} + \frac{1}{2}\,x^2\mathrm{D}^2 + \ldots)\,\phi\,(a, b \ldots).$$

The "remainder after n terms" is, of course, $\frac{1}{n!}\,x^n\mathrm{D}^n\phi\,(fx_1, f'x_1 \ldots)$, where $x_1 = \theta x$, θ being some proper fraction.

The most important special case of the general theorem is this,

$$\phi fx = \phi a + x\mathrm{D}\phi a + \frac{1}{2}\,x^2\mathrm{D}^2\phi a + \frac{1}{2\,.\,3}\,x^3\mathrm{D}^3\phi a + \ldots \qquad (3)$$

Here, of course, as before,

$$fx = a + bx + \frac{1}{2}\,cx^2 + \frac{1}{2\,.\,3}\,dx^3 + \ldots,$$

$$\mathrm{D} = b\,\frac{d}{da} + c\,\frac{d}{db} + \ldots.$$

This formula will be found, I think, superior in directness to Arbogast's method, while yielding the same results for the same expenditure of labor.

If, in (3), we put $b = 1$, and $c = d = \ldots = 0$, we have remaining, as a special case, Taylor's theorem.

Both (1) and (3) may be modified as follows. For a, write α; for b, β; for c, 2γ; for d, 2.3δ; and so on. Then

$$\begin{aligned} fx &= \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots, \\ f'x &= \beta + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3 + \ldots, \\ f''x &= 2\gamma + 2.3\delta x + 3.4\varepsilon x^2 + 4.5\zeta x^3 + \ldots, \\ f'''x &= 2.3\delta + 2.3.4\varepsilon x + 3.4.5\zeta x^2 + \ldots. \end{aligned}$$

and so on, and
$$\mathrm{D} = \beta\frac{d}{d\alpha} + 2\gamma\frac{d}{d\beta} + 3\delta\frac{d}{d\gamma} + 4\varepsilon\frac{d}{d\delta} + \ldots.$$

Let
$$\psi(fx, f_1x, f_2x \ldots) = \phi(fx, f'x, f''x \ldots),$$

where

$$\begin{aligned} fx &= \alpha + \beta x + \gamma x^2 + \delta x^3 + \ldots, \\ f_1x &= f'x = \beta + 2\gamma x + 3\delta x^2 + \ldots, \\ f_2x &= \frac{1}{2}f''x = \gamma + 3\delta x + 6\varepsilon x^2 + \ldots, \\ f_3x &= \frac{1}{2.3}f'''x = \delta + 4\varepsilon x + 10\zeta x^2 + \ldots, \end{aligned}$$

and so on. The law of the numerical coefficients is obvious. Each may be found by adding the one above to the one on the left hand. Then (1) assumes this form,

$$\psi(fx, f_1x, f_2x \ldots) = (1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \ldots)\,\psi(\alpha, \beta, \gamma \ldots), \qquad (4)$$

and (3) becomes

$$\psi fx = \psi\alpha + x\mathrm{D}\psi\alpha + \frac{1}{2}x^2\mathrm{D}^2\psi\alpha + \frac{1}{2.3}x^3\mathrm{D}^3\psi\alpha + \ldots. \qquad (5)$$

In these theorems, as has just been stated,

$$\mathrm{D} = \beta\frac{d}{d\alpha} + 2\gamma\frac{d}{d\beta} + 3\delta\frac{d}{d\gamma} + \ldots.$$

Occasions may arise when these modifications may be found more convenient in use than the forms first given.

The formulæ thus far considered hold good, not only when fx is an algebraic function, but also in cases where fx is an infinite series. Let us now assume that fx contains no power of x greater than x^n; in other words, let

$$fx = a + bx + \ldots + \frac{1}{n-1!}jx^{n-1} + \frac{1}{n!}kx^n.$$

Having thus restricted the meaning of fx, let a be replaced by α; b by $n\beta$; c by $n^{2)}\gamma$; d by $n^{3)}\delta$; and so on, where $n^{r)}$ represents the factorial $n(n-1)(n-2)\ldots(n-r+1)$. Then

$$
\begin{aligned}
fx &= \alpha + n\beta x + \frac{1}{2} n^{2)}\gamma x^2 + \ldots + \varkappa x^n, \\
f'x &= n\beta + n^{2)}\gamma x + \frac{1}{2} n^{3)}\delta x^2 + \ldots + n\varkappa x^{n-1}, \\
f''x &= n^{2)}\gamma + n^{3)}\delta x + \ldots + n^{2)}\varkappa x^{n-2}, \\
&\ldots\ldots\ldots\ldots \\
f^{(n-1)}x &= n^{n-1)}\iota + n^{n)}\varkappa x, \\
f^{(n)}x &= n^{n)}\varkappa.
\end{aligned}
$$

Let

$$\psi(fx, f_1x, \ldots f_{n-1}x, f_nx) = \phi(fx, f'x, \ldots f^{(n-1)}x, f^{(n)}x),$$

where

$$
\begin{aligned}
fx &= \alpha + n\beta x + \frac{1}{2} n^{2)}\gamma x^2 + \ldots + \varkappa x^n, \\
f_1x &= \frac{1}{n} f'x = \beta + (n-1)\gamma x + \frac{1}{2}(n-1)^{2)}\delta x^2 + \ldots + \varkappa x^{n-1}, \\
f_2x &= \frac{1}{n^{2)}} f''x = \gamma + (n-2)\delta x + \frac{1}{2}(n-2)^{2)}\varepsilon x^2 + \ldots + \varkappa x^{n-2}, \\
&\ldots\ldots\ldots\ldots \\
f_{n-1}x &= \frac{1}{n^{n-1)}} f^{(n-1)}x = \iota + \varkappa x, \\
f_nx &= \frac{1}{n^{n)}} f^{(n)}x + \varkappa.
\end{aligned}
$$

Then

$$\psi(\alpha, \beta, \gamma \ldots \iota, \varkappa) = \phi(a, b, c \ldots j, k);$$

and, since $\frac{d}{da} = \frac{d}{d\alpha}$, $\frac{d}{db} = \frac{d}{dn\beta} = \frac{1}{n}\frac{d}{d\beta}$, and so on,

$$\mathrm{D} = n\beta\frac{d}{d\alpha} + (n-1)\gamma\frac{d}{d\beta} + (n-2)\delta\frac{d}{d\gamma} + \ldots + 2\iota\frac{d}{d\theta} + \varkappa\frac{d}{d\iota}.$$

Making these several substitutions in (1), we derive a known result, due to Dr. Salmon,

$$\psi(fx, f_1x, \ldots f_{n-1}x, f_nx) = (1 + x\mathrm{D} + \frac{1}{2}x^2\mathrm{D}^2 + \ldots)\,\psi(\alpha, \beta, \ldots \iota, \varkappa). \quad (6)$$

This theorem, to which the name of Salmon's Theorem * has been assigned, is usually given in the opposite order, as, for example, in the following statement: † "If f is any function of $a, b, c\ldots$, and f_1 the same function of

* This name was assigned by me, without consulting Dr. Salmon, in error. The theorem, as far as I have been able to ascertain, seems first to have been stated and proved in a complete form by Chev. Faà de Bruno. See his valuable *thesaurus* "Théorie des formes binaires," (Turin, 1876,) p. 130, and again p. 310. So much of it as is contained in Salmon's classic work had been previously stated by Cayley in Crelle. J. J. S.

† See Salmon, *Higher Algebra*, 3d ed., page 59, art. 63; Sylvester, "On a Generalization of Taylor's Theorem," *Philosophical Magazine*, Aug., 1877; *Jahrbuch über die Fortschritte der Mathematik*, vol. 9, p. 171.

$a_1, b_1, c_1 \ldots$, where $a_1 = a$, $b_1 = b + ah$, $c_1 = c + 2bh + ah^2$, $d_1 = d + 3ch + 3bh^2 + ah^3 \ldots$, and if Ω represents the operation $a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \ldots$, then

$$f_1 = f + h\Omega f + \frac{1}{2}h^2\Omega^2 f + \frac{1}{2\,.\,3}h^3\Omega^3 f + \ldots\text{''}$$

Here, of course, h stands for our x, Ω for our D, a for our $\varkappa$, b for our ι, and so on.

Symbolic demonstrations of these theorems are not difficult to obtain. Thus, let u_y be an arbitrary function of y. Employing the symbol of enlargement E^n, representing an operation such that $\mathrm{E}^n\phi y = \phi(y + n)$, we have $u_1 = \mathrm{E}u_0$, $u_2 = \mathrm{E}^2u_0$, and so on. Let ψ be any function of $u_0, \mathrm{E}u_0, \mathrm{E}^2u_0 \ldots$, and in this function let us treat the symbol E, wherever it occurs, as itself the subject of operation. Employing a slightly different type, let E represent the operation of enlargement with respect to E, so that $\mathsf{E}^x\phi\mathrm{E} = \phi(\mathrm{E} + x)$, and, similarly, let $\mathsf{D} = \log \mathsf{E} = \frac{d}{d\mathrm{E}}$, representing differentiation with respect to E. Then, since

$$\mathsf{E}^x = 1 + x\mathsf{D} + \frac{1}{2}x^2\mathsf{D}^2 + \frac{1}{2\,.\,3}x^3\mathsf{D}^3 + \ldots,$$

$$\mathsf{E}^x\psi = \psi + x\mathsf{D}\psi + \frac{1}{2}x^2\mathsf{D}^2\psi + \frac{1}{2\,.\,3}x^3\mathsf{D}^3\psi + \ldots.$$

It will be seen, on examination, that this is Salmon's theorem. The operation E^x changes E, wherever it occurs in ψ, into $\mathrm{E} + x$, and ψ becomes the same function of $u_0, (\mathrm{E} + x)u_0, (\mathrm{E} + x)^2u_0, \ldots$ that it was before of $u_0, \mathrm{E}u_0, \mathrm{E}^2u_0 \ldots$ That is to say,

u_0 is replaced by u_0,
$u_1 = \mathrm{E}u_0$ " " " $(\mathrm{E} + x)\,u_0 = u_1 + xu_0$,
$u_2 = \mathrm{E}^2u_0$ " " " $(\mathrm{E} + x)^2\,u_0 = u_2 + 2xu_1 + x^2u_0$,

and so on. Let $\phi_n(\mathrm{E}^nu_0)$ express ψ as a function of E^nu_0, all the other variables involved in ψ being regarded as constant. Then*

$$\mathsf{D}\psi = \mathsf{D}\phi_1(\mathrm{E}u_0) + \mathsf{D}\phi_2(\mathrm{E}^2u_0) + \ldots.$$

Since

$$\mathsf{D}\phi_n(\mathrm{E}^nu_0) = \phi'_n(\mathrm{E}^nu_0)\,\mathsf{D}\mathrm{E}^nu_0 = \phi'_n(\mathrm{E}^nu_0)\,n\mathrm{E}^{n-1}u_0 = nu_{n-1}\frac{d\psi}{du_n},$$

*It is unnecessary to discuss the well-known principles by which the legitimacy of these steps is established. Any reader to whom they are not familiar may refer to the "Essay on the Calculus of Enlargement" recently printed in this Journal.

we have, after substitution,

$$\mathrm{D}\psi = u_0 \frac{d\psi}{du_1} + 2u_1 \frac{d\psi}{du_2} + 3u_2 \frac{d\psi}{du_3} + \ldots;$$

or, since ψ may be any function of the variables in question,

$$\mathrm{D} = u_0 \frac{d}{du_1} + 2u_1 \frac{d}{du_2} + 3u_2 \frac{d}{du_3} + \ldots.$$

Or, similarly, let $\mathrm{D} = \frac{d}{d0}$, and let

$$v_z = u_0 + u_1 z + \frac{1}{2} u_2 z^2 + \frac{1}{2\,.\,3} u_3 z^3 + \ldots,$$

then $u_0 = v_0$, $u_1 = \mathrm{D}v_0$, $u_2 = \mathrm{D}^2 v_0$, and so on, and if we make E such that $\mathsf{E}^x \phi \mathrm{D} = \phi(\mathrm{D} + x)$, and $\mathsf{D} = \log \mathsf{E} = \frac{d}{d\mathrm{D}}$, the demonstration will proceed as before. If, on the other hand,

$$v_z = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \ldots,$$

we shall, proceeding as in the last case, obtain a theorem equivalent to (1), provided fx is an algebraic function.

Preliminary Notes on Mr. Hall's Recent Discovery.

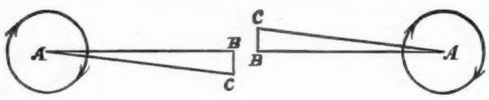
By H. A. Rowland.

The recent discovery by Mr. Hall of a new action of magnetism on electric currents opens a wide field for the mathematician, seeing that we must now regard most of the equations which we have hitherto used in electromagnetism as only approximate, and as applying only to some ideal substance which may or may not exist in nature, but which certainly does not include the ordinary metals. But as the effect is very small, probably it will always be treated as a correction to the ordinary equations.

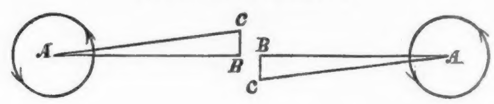
The facts of the case seem to be as follows, as nearly as they have yet been determined: Whenever a substance transmitting an electric current is placed in a magnetic field, besides the ordinary electromotive force in the medium, we now have another acting at right angles to the current and to the magnetic lines of force. Whether there may not be also an electromotive force in the direction of the current has not yet been determined with accuracy, but it has been proved within the limits of accuracy of the experiment that no electromotive force exists in the direction of the lines of magnetic force. This electromotive force in a given medium is proportional to the strength of the current and to the magnetic intensity, and is reversed when either the primary current or the magnetism is reversed. It has also been lately found that the direction is different in iron from what it is in gold or silver.

To analyze the phenomenon in gold, let us suppose that the line AB represents the original current at the point A and that BC is the new effect. The magnetic pole is supposed to be either above or below the paper as the case may be. The line AC will represent the final resultant electromotive force at the point A. The circle with arrow represents the direction in which the current is rotated by the magnetism.

North Pole Above.



North Pole Below.



It is seen that all these effects are such as would happen were the electric current to be rotated in a fixed direction with respect to the lines of magnetic force and to an amount depending only on the magnetic force and not on the current. This fact seems to point immediately to that other very important case of rotation, namely, the rotation of the plane of polarization of light. For, by Maxwell's theory, light is an electrical phenomenon and consists of waves of electrical displacement, the currents of displacement being at right angles to the direction of propagation of the light. If the action we are now considering takes place in dielectrics, which point Mr. Hall is now investigating, the rotation of the plane of polarization of light is explained.

I give the following very imperfect theory at this stage of the paper, hoping to finally give a more perfect one either in this paper or a later one.

Let $\mathfrak{H}$ be the intensity of the magnetic field, and let E be the original electromotive force at any point, and let $\mathfrak{c}$ be a constant for the given medium. Then the new electromotive force, E', will be

$$E' = \mathfrak{c}\,\mathfrak{H}\,E.$$

and the final electromotive force will be rotated through an angle which will be very nearly equal to $\mathfrak{c}\,\mathfrak{H}$. As the wave progresses through the medium, each time it, the electromotive force, is reversed it will be rotated through this angle, so that the total rotation will be this quantity multiplied by the number of waves. If λ is the wave length in air and i is the index of refraction and c is the length of medium, then the number of waves will be $\frac{ci}{\lambda}$ and the total rotation

$$\theta = c\,\mathfrak{c}\,\mathfrak{H}\frac{i}{\lambda}.$$

The direction of rotation is the same in diamagnetic and feromagnetic bodies as we find by experiment, being different in the two; for it is well known that the rotation of the plane of polarization is opposite in the two media, and Mr. Hall now finds *his* effect to be opposite in the two media. This result I anticipated from this theory of the magnetic rotation of light.

But the formula makes the rotation inversely proportional to the wave length, whereas we find it more nearly as the square or cube. This I consider to be a defect due to the imperfect theory, and it would possibly disappear from the complete dynamical theory. But the formula at least makes the rotation increase as the wave length decreases, which is according to experiment. Should an exact formula be finally obtained, it seems to me

that it would constitute a very important link in the proof of Maxwell's theory of light, and, together with a very exact measure of the ratio of the electromagnetic to the electrostatic units of electricity which we made here last year, will raise the theory almost to a demonstrated fact. The determination of the ratio will be published shortly, but I may say here that the final result will not vary much, when all the corrections have been applied, from 299,700,000 metres per second, and this is almost exactly the velocity of light. We cannot but lament that the great author of this modern theory of light is not now here to work up this new confirmation of his theory, and that it is left for so much weaker hands.

But before we can say definitely that this action explains the rotation of the plane of polarization of light, the action must be extended to dielectrics, and it must be proved that the lines of electrostatic action are rotated around the lines of force as well as the electric currents. Mr. Hall is about to try an experiment of this nature.

I am now writing the full mathematical theory of the new action, and hope to there consider the full consequences of the new discovery.

On Certain Ternary Cubic-Form Equations.

By J. J. Sylvester.

Excursus A. *On the Divisors of Cyclotomic Functions.*

Title 1. *Cyclotomic Functions of the 1st Species.* In the preceding section which should have been termed and will be hereafter referred to as the *Proem* of Chapter I, I stated that the proof of the first batch of theorems on the irresoluble cases of equations in numbers of the form $x^3+y^3+Az^3=0$, or, as we might say, of the forms of numbers A irresoluble into a pair of rational cubes, depends on the demonstration of the form of the numerical linear divisors of the function x^3-3x+1. At the time when this proem went to press I had reduced to a certainty the law of the divisors by numerical verifications without end, but had not obtained a rational demonstration of it, nor was I able to find such or even a statement of the law itself in any of the current text-books, such as Gauss, Legendre, Bachmann, Lejeune-Dirichlet or Serret. I was therefore compelled to seek out a demonstration for myself, and in so doing was unavoidably led to consider the general theory of the species of *cyclotomic* (*Kreistheilung*) functions of which the cubic functions above written is an example of what may be called the second species and incidentally also the theory of the simpler or first species which, although discussed ever since the time of Euler, appears to me to remain still in a somewhat cloudy and incomplete condition. As this inquiry extends beyond the strict needs of the subject which called it forth, I entitle it an *excursus.* It will be necessary for me eventually to introduce another and still more important excursus or lateral digression on certain consequences of the Geometrical Theory of Residuation, which theory itself also took its rise in and is required for the purposes of the arithmetical theory which forms the subject of the entire memoir.

If fx is any rational integral function of the order ω in its variable, we know that in respect to a prime number p as modulus fx regarded as the subject of a congruence cannot have more than ω distinct real roots. If we take p^j as modulus, certain conditions increasing in number with the value of j, will have to be satisfied in order that fx may have a superfluity (*i. e.* more than ω) of real roots.

One condition, the universal *sine quâ non*, will serve for the object I have in view, so that it will be sufficient to make $j=2$. Obviously when this superfluity exists two of the roots must differ by a multiple of p since otherwise there would be a superfluity of roots *quâ* the first power of p as modulus. If then a and $a+\lambda p$ where $\lambda < p$ be two of the roots, we have $fa \equiv 0$ and $fa + \lambda f'a \,.\, p + Rp^2 \equiv 0$ mod. p^2. Hence $fa \equiv 0$ and $f'a \equiv 0$ mod. p, so that $fa + \lambda p = 0$ and $f'a + \mu p = 0$.

Applying the dialytic method to eliminate a it is obvious that the resultant of these two equations will differ only by a multiple of p from that of fa and $f'a$, *i. e.* from the arithmetical discriminant of fa (I use the term arithmetical to distinguish it from the algebraical discriminant in obtaining which latter fx is supposed to be affected with binomial numerical coefficients ω, $\frac{\omega \,.\, \omega - 1}{2}$, ... and the factor ω to be struck out from each of the two equations $\frac{df(x,1)}{dx} = 0$, $\frac{df(x,1)}{d1} = 0$).

We see then that a rational integer function (the subject of a congruence) cannot have a superfluity of roots in respect to the power of a prime p^j as modulus, unless the strict (arithmetical) discriminant of the function contains p.

It is necessary for the purpose I have in view to express the strict relation between the arithmetical discriminant of a function Δfx and the product of the squares of the differences of its roots $\zeta^2 fx$. I shall for greater simplicity suppose that the initial coefficient of fx is unity, as it is in the cases with which we shall have to deal.

We know that $\Delta f = \mu \zeta^2 f$ where μ is a function of n the order of f, so that to determine μ we may specialize f in any manner we please, provided the order is maintained. Let $fx = x^n - 1$. Then it is easily proved that, making $\rho = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$,

$$(-)^{n\frac{n-1}{2}} \zeta^2 f = (\rho^{n-1})^{(n-1)\frac{n}{2}} \,.\, n^n,$$

so that
$$\zeta^2 f = (-)^{\frac{(n-1)(n-2)}{2}} \,.\, n^n,$$

and
$$\Delta f = (-)^{n-1} \,.\, n^{2n-2}.$$

Hence
$$\Delta f = (-)^{(n-1)\frac{n}{2}} \,.\, n^{n-2} \zeta^2 f\text{*}$$

*As regards the application to be made of this result it was of course not necessary to determine the index of the power to which $(-)$ is raised, but it was hardly worth while to leave it undetermined.

expresses the universal relation between the arithmetical discriminant and the squared product of the root-difference of a function. If we had been dealing with the algebraical discriminant, it would have been necessary to replace n^{n-2} by n^{-n} in the above equation. It is furthermore to be observed that the discriminant is fixed in its sign by the condition that the term containing the highest power of the product of the expressed coefficients is to be taken positively.

So again it will be seen presently to be necessary to ascertain the strict relation between the resultant of two functions of degrees r, s and the product of the differences between the several roots ρ of the one and the several roots σ of the other of them, or, as we may say, between $R_{r,s}$ and $D_{\rho,\sigma}$, where if we choose to pay attention to algebraical signs that of $R_{r,s}$ may be understood to mean the resultant so taken that the term containing the highest power of the coefficient in the r-degreed function is positive and $D_{\rho,\sigma}$ to mean the product of the rs differences $(\rho - \sigma)$.

I shall again, for greater simplicity, suppose the initial coefficients of each of the two functions to be unity.

We know that $R_{r,s} = \mu D_{\rho,\sigma}$ where μ is a function of r and s exclusively. To determine it we may take x^r and $x^s + 1$ as the two functions, it will be found without difficulty that

$$R_{r,s} = 1^* \text{ and } D_{\rho,\sigma} = \left(-(-1)^{\frac{1}{s}}\right)^{rs} = (-)^{rs+r}.$$

Hence we have universally $R_{r,s} = (-)^{rs+r} D_{\rho,\sigma}$.

This seems to be the proper place to ascertain (what will be needed for future purposes) how far or under what qualifications the reciprocal connexion of the two facts: 1. Of two functions in x having a common root. 2. Of their resultant being zero, admits of being extended to roots of congruences in respect to a prime-number modulus.

Suppose fx, gx to be two in all respects (numerically† as well as algebraically) integer rational functions of the degrees i, j in x, then by eliminating dialytically $(i + j - 1)$ powers of x between

$$fx,\ xfx,\ x^2fx \ldots x^{j-1}fx,\quad gx,\ xgx,\ \ldots x^{i-1}gx,$$

---

* Thus *ex gr.* let $r = 4$, $s = 2$. Then $R_{r,s}$ is the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +\,x^3 & & & \\
 & x^4 & & +\,x^2 & & \\
 & & x^3 & & +\,x & \\
 & & & x^2 & & +\,1
\end{array}$$

which is obviously equal to unity.

† By which I mean that the coefficients are exclusively integer numbers.

we may obtain the equation $\lambda x fx + \mu x gx = Rx^q$ (q having any integer value from 0 to $i+j-1$) where R is the resultant of f, g and λx, μx are in all respects integer functions of x of degrees $j-1$ and $i-1$ in x whose values depend on the value of q. If, then, fx and gx are simultaneously zero for some value of x, we must universally have $R=0$ even if x should be zero, for thus we might make $q=0$.

But this equation will not suffice to show that fx and gx will simultaneously vanish for some value of x, provided that $R=0$ for every value of x which makes fx vanish, might, as far as this equation discloses, (and for all values of g), have the effect of making μx vanish.* We may, however, prove the fact in question, on a certain hypothesis to be presently stated, by availing ourselves of the knowledge that R is, to a *numerical factor près*, the product of the differences between the roots of f and those of g.

The hypothesis I make is that $fx \equiv 0$ mod. p is a congruence *all whose roots are real;* in this case I shall show that if the resultant R of fx and gx satisfies the congruence $R \equiv 0$ mod. p (*i. e.* if R contains p) then gx must have at least one real root in common with fx *quâ* modulus p.

From the congruence of $fx \equiv 0$ mod. p we may, by a well known principle, infer the existence of an equation $Fx = fx + p\phi x = 0$ whose roots are the same as those of the congruence above written, and the dialytic method of elimination renders it self-evident that the resultant of Fx and gx will differ only by a multiple of p from that of fx and gx, and will, therefore, be a multiple of p.

If, then, we call the roots of Fx (all real by hypothesis) $a_1, a_2, \ldots a_i$, we shall have $ga_1 . ga_2 . ga_3 \ldots ga_i \equiv 0$ mod. p, and, as all the factors on the left hand side of the equation are real, one of them must contain p. Hence, if $R(fx, gx) \equiv 0$ mod. p, and $fx \equiv 0$ mod. p *has all its roots real,* one of these roots must belong also to the congruence $gx \equiv 0$ mod. p.

Going back now to what precedes this investigation, let us determine strictly the relation between the arithmetical discriminants and resultant of two functions in x and the discriminant of their prodnct.

Let ω, ω_1 be the degrees in x of two altogether integer functions fx, f_1x, and suppose $Fx = fx . f_1x$. Then obviously $\zeta^2 Fx = \zeta^2 fx . \zeta^2 f_1 x . (D(fx, f_1x))^2$. Hence $\omega^{\omega-2} . \omega_1^{\omega_1-2} \Delta Fx = (\omega+\omega_1)^{\omega+\omega_1-2} f\Delta x . \Delta f_1 x (R(fx, f_1x))^2$.

* I think it would not be incorrect to say that *in all cases* the fact of the resultant of two functions of x containing a prime number raises a strong presumption that the functions have a common congruence root in respect to that number.

If, then, p any prime number is contained in Δfx, and ω, ω_1 are each less than p, p will necessarily be contained in ΔFx. And as a particular case of this theorem, if p were contained in the discriminant of any factor of $x^{p-1}-1$ it would be contained in the discriminant of $x^{p-1}-1$, *i. e.* in a power of $(p-1)$, which is impossible. Hence, by a preceding theorem, no factor of $x^{p-1}-1$, regarded as the subject of a congruence, can contain *a superfluity* of real roots (*i. e.* more real roots than there are units in its degree) in respect to the modulus p^j.

It is easy to show, although I do not find it distinctly stated in any of the current text-books, that $x^{p-1}-1\equiv 0$ mod. p^j has $p-1$ real roots.

For let $x=y^{p^{j-1}}$. Then the congruence becomes

$$y^{p^{j-1}.(p-1)}-1\equiv 0 \text{ mod. } p^j,$$

where $p^{j-1}.(p-1)$ is what is commonly designated as the ϕ function of p^j, the number of numbers less than p^j and prime to it, (the so-called ϕ function of any number I shall here and hereafter designate as its τ function and call its Totient). This last congruence by Fermat's extended theorem has all its roots real. It is easy to see that they will consist of $(p-1)$ groups, each group containing p^{j-1} numbers for which the value of x *quâ* modulus p^j will be the same, but different for numbers belonging to two different groups. For let y_1 be any of the y roots, and $y_2^{p^{j-1}}-y_1^{p^{j-1}}\equiv 0$ mod. p^j. Then *quâ* mod. p, $y_2^{p^{j-1}}\equiv y_2$ and $y_1^{p^{j-1}}\equiv y_1$, because $p^{j-1}-1$ contains $p-1$.

All the values of y_2 will, therefore, be comprised in the series

$$y_1,\ y_1+p,\ y_1+2p,\ \ldots\ y_1+(p^{j-1}-1)\,p,$$

and $$(y_1+\lambda p)^{p^{j-1}}=y_1^{p^{j-1}}+p^{p^j}.\,Q.$$

Hence the p^j terms of the series (and no other values of z) all satisfy the congruence

$$z^{p^{j-1}}-y_1^{p^{j-1}}\equiv 0 \text{ mod. } p^j.$$

Hence $x=y^{p^{j-1}}$ has $(p-1)$ distinct real values *quâ* p^j or there are $(p-1)$ real roots to the congruence $x^{p-1}-1\equiv 0$ mod. p^j. Hence, if fx is any factor of $x^{p-1}-1$, $fx\equiv 0$ [mod. p^j] will have all its roots real.

For let $fx.f_1x=x^{p-1}-1$.

Then since $x^{p-1}\equiv 0$ mod. p^j has all its roots real, and fx and f_1x have no congruence root *quâ* mod. p in common* if $fx\equiv 0$ to the modulus p^j has not *its full quota*, f_1x will have a *superfluity* of roots, but this has been shown to be impossible.

* For if this were the case two factors of $x^{p-1}-1$ *quâ* mod. p having two roots in common $x^{p-1}-1$ would not have its full quota of roots.

Now, let $p = mk + 1$. Then $x^k - 1$ is a factor of $x^{p-1} - 1$. Let $\chi_k x$ be the factor of $x^k - 1$, which contains all its primitive roots; this is what I term a *cyclotomic function of the first species* to the index k. $\chi_k x$ being a factor of $x^k - 1$ is a factor of $x^{p-1} - 1$, and will therefore, by what has just been shown, have all its roots real *quâ* the modulus p^j.

Hence a cyclotomic function of the 1st species to the index k contains, as a divisor, any power of any prime number of the form $mk + 1$, and, moreover, if ω is its degree, (where ω represents the *totient* of k), $(mk + 1)^j$ will be an ω-fold divisor of the function, *i. e.* will be a divisor thereof corresponding to ω distinct values of the variable of the function, *i. e.* values incongruent with one another *quâ* the modulus p^j.

The divisors of the cyclotomic function to index h may be divided into two classes, viz: divisors which do not divide the index, which may be called superior or extrinsic divisors, and divisors which divide at the same time the function and its index which may be termed inferior or intrinsic divisors. I shall begin with showing that any prime number extrinsic divisor diminished by unity must contain the index, *i. e.*, that if p is an extrinsic divisor and k the index, we must have $p = mk + 1$ which is a reciprocal proposition to the one just established.

If possible let p, any prime such that $p - 1$ does not contain k nor k contain p, be a divisor of the cyclotomic function of the first species $\chi_k x$. And let δ be the greatest common divisor of $p - 1$ and k. Then we shall have $x^\delta - 1 \equiv 0$ mod. p. But we have also $\chi_k x \equiv 0$ mod. p. Hence the resultant of $x^\delta - 1$ and $\chi_k x$ must contain p, but $\frac{x_k - 1}{x^\delta - 1}$ contains $\chi_k x$; *à fortiori* therefore the resultant of this and $x^\delta - 1$ will contain p. But this resultant is evidently equal to the value of $\frac{x^k - 1}{x^\delta - 1}$ (where $x^\delta = 1$) raised to the power δ, *i. e.* $= \left(\frac{k}{\delta}\right)^\delta$ and therefore, *ex-hypothesi*, does not contain p.

It has thus been proved that every extrinsic divisor of $\chi_k x$ can only be of the form $mk + 1$.

Next let $k = k_1 p^j$ (k_1 being prime to p) and suppose p to be a divisor of $\chi_k x$.

Then p is a divisor of $(x^{p^j})^{k_1} - 1$ and, therefore, by what has been shown, must be of the form $mk_1 + 1$, unless $x^{p^j} - 1$ contained p in which case since $p^j - 1$ is divisible by $p - 1$, $x - 1$ must contain p and consequently p will be a divisor of $\chi_k 1$.

To find the value of $\chi_k 1$ we may proceed as follows:

Let $k = a^\alpha . b^\beta . c^\gamma . d^\delta . e^\varepsilon$. Then the totient of k is

$$a^{\alpha-1} . b^{\beta-1} . c^{\gamma-1} . d^{\delta-1} . e^{\varepsilon-1} \left\{ \begin{array}{l} \alpha\beta\gamma\delta\varepsilon + \Sigma\alpha\beta\gamma + \Sigma\alpha \\ - \Sigma\alpha\beta\gamma\delta - \Sigma\alpha\beta - 1 \end{array} \right\},$$

and if we write this $L + M + N - P - Q - R$

$$\chi_k x = \frac{(x^L - 1)(x^M - 1)(x^N - 1)}{(x^P - 1)(x^Q - 1)(x^R - 1)},$$

and so in general the expression for $\chi_k x$, however many the distinct prime factors of k, imitates and follows *pari passu* the expression for the totient of k; and if $L, M, N, \ldots$ be the positive terms and $P, Q, R, \ldots$ be the negative ones in the algebraical representation of that totient by the common theory of vanishing fractions, shows that $\chi_k 1 = \frac{L . M . N \ldots}{P . Q . R \ldots}$. There are two cases:

1°. When k contains i distinct prime factors, where $i > 1$. In that case supposing a to be one of them and α its index, the index of a in $L . M . N \ldots$ will be

$$\alpha \left\{ 1 + \frac{(i-1)(i-2)}{1.2} + \frac{(i-1)(i-2)(i-3)(i-4)}{1.2.3.4} + \ldots \right\}$$

and in $P . Q . R \ldots .$

$$\alpha \left\{ (i-1) + \frac{(i-1)(i-2)(i-3)}{1.2.3} \ldots \right\},$$

so that the index in the quotient is $\alpha (1-1)^{i-1}$, *i. e.* is zero. And so for $b, c, \ldots .$ Hence $\chi_k 1 = 1$.

2°. When $i = 1$ and $k = a^\alpha$, the value of $\chi_k x = \frac{x^{a^\alpha} - 1}{x^{a^{\alpha-1}} - 1}$, and consequently $\chi_k 1 = a$. Hence, when $k = k_1 p^j$, and k_1 is not unity, p, if a divisor of $\chi_k x$, must be of the form $mk_1 + 1$. Moreover, the case of $k_1 = 1$ offers no exception to this conclusion, inasmuch as when $k_1 = 1$, p, (like every other number) comes under the form $mk_1 + 1$.

It now remains to show the converse that if $k = k_1^{p^j}$ and $p = mk_1 + 1$, p will be a divisor of $\chi_k x$.

For the sake of greater simplicity, we may consider apart the case where $k = p^j$. Here $\chi_k x = \frac{x^{p^j} - 1}{x^{p^{j-1}} - 1} = 1 + x^{p^{j-1}} + x^{2p^{j-1}} + \ldots + x^{(p-1)p^{j-1}}$, which, (to modulus p) $\equiv 1 + x + x^2 + \ldots . + x^{p-1} \equiv \frac{x^p - 1}{x - 1}$, and, therefore, can only contain p, if $x^p - 1$, and, consequently, $x - 1$ contains it. Hence, the only root of $\chi_k x \equiv 0$ [mod. p], for this case is $x = 1$.

Moreover, only p itself, and no higher power of p, can be a divisor of the cyclotomic function in question, because

$$\frac{(1+\lambda p)^{p^j}-1}{(1+\lambda p)^{p^{j-1}}-1} = \frac{\lambda p^{j+1}+\ldots}{\lambda p^j+\ldots} = p + Bp^2 + Cp^3 + \ldots + Lp^{(p-1)p^{j-1}}$$

does not contain p^2.*

To save unnecessary fatigue of attention, about a small matter, to my readers and myself, I will take, as a representative of the general case, $k = k_1 p$, $k_1 = abc$, $p = mk_1 + 1$; it will easily be verified that the increase of the number of distinct prime factors a, b, c, or the affection of them or of p with indices, will in no manner affect the course of the demonstration or the validity of the conclusion.

In the above *special case*

$$\chi_k x = \frac{(x^{abcp}-1)(x^{ab}-1)(x^{ac}-1)(x^{bc}-1)(x^{ap}-1)(x^{bp}-1)(x^{cp}-1)(x-1)}{(x^{abc}-1)(x^{abp}-1)(x^{acp}-1)(x^{bcp}-1)(x^{a}-1)(x^{b}-1)(x^{c}-1)(x^{p}-1)}.$$

Let now $x^{k_1}-1=0$, so that $x^p = x$. Then obviously $\chi_k x = \dfrac{x^{abcp}-1}{x^{abc}-1} = p$. Hence the resultant of $\chi_{k_1} x$ and $\chi_k x$ is $p^{\tau(k_1)}$ (τk_1 meaning the totient of k_1). Consequently since $\chi_{k_1} x \equiv 0$ [mod. p] has all its roots real, one root at least of $\chi_k x \equiv 0$ [mod. p] will be a root of the preceding congruence.

It will be noticed that if instead of $\chi_{k_1} x$ we took $\chi_{k'_1} x$ where k'_1 is a factor of k_1 it would not be true that the resultant of it and $\chi_k x$ would contain p.

For example, if $k'_1 = ab$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp+1}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1} = \frac{p}{p} = 1.$$

Or again if $k'_1 = a$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}\cdot\frac{x^{ac}-1}{x^{acp}-1}\cdot\frac{x^{ap}-1}{x^{a}-1} = p_1\frac{1}{p}\cdot\frac{1}{p}\cdot p = 1$$

as before. So that the resultant instead of being p would, in each case, be 1, and consequently $x^k-1\equiv 0$ [mod. p] and $x^{k'_1}-1\equiv 0$ [mod. p] could not have a root in common. And so in general it may be shown that if $k = k_1 p^j$ and $k'_1 = \dfrac{k_1}{\delta}$ the resultant of $x^{k'_1}-1$ and $\chi_k x$ is 1, except when $\delta = 1$ in which case it is p.

Hence the roots of $\chi_k x \equiv 0$ [mod. p] are to be sought not among all the roots of $x^{k_1}-1\equiv 0$ [mod. p], but exclusively among only such of them as belong to the congruence $\chi_{k_1} x \equiv 0$ [mod. p].

* When $p = 2$ and $j = 1$ the third term will not be of a higher power in p than the second term in the development of the numerator, so that the conclusion ceases to hold; as ought to be the case for the cyclotomic of the 1st species to the index 2, viz: $x + 1$ will obviously contain every power of 2 as a divisor.

We have seen that if p, a prime number, is an extrinsic divisor of a cyclotomic function to the index k, any power of p is also a divisor of the function. On the contrary, if p is an intrinsic divisor it will appear that p^2 cannot (and consequently no higher power of p than the 1st, can) be a divisor. For if x satisfies the congruence $\chi_{k_1}x \equiv 0$ [mod. p] we must have $x^{k_1} = 1 + \lambda p$ and $x^p = x^{mk_1}.x = (1 + mp)\,x$, where m represents a series of ascending powers of p. Hence

$$\chi_k x = \frac{x^{k_1 p} - 1}{\chi^{k_1} - 1} \cdot \frac{x^{ab} - 1}{x^{abp} - 1} \cdot \frac{x^{ac} - 1}{x^{acp} - 1} \cdot \frac{x^{bc} - 1}{x^{bcp} - 1} \cdot \frac{x^{ap} - 1}{x^{a} - 1} \cdots,$$

where the first factor, being equal to $x^{k_1(p-1)} + x^{k_1(p-2)} + \ldots + 1$, will be of the form $p\,(1 + Pp)$, P being a series containing only positive powers of p. Again,

$$\frac{x^{ab} - 1}{(1 + Qp)\,x^{ab} - 1} = 1 + \frac{Qpx^{ab}}{1 - x^{ab}} + \frac{Q^2p^2x^{2ab}}{(1 + x^{ab})^2} + \ldots = 1 + Q_1 p$$

where Q_1 is an infinite series containing positive powers only of p and x.

In like manner $\dfrac{x^{ap} - 1}{x^{a} - 1} = \dfrac{(1 + Rp)\,x^{a} - 1}{x^{a} - 1} = 1 + R_1 p$ where R_1 (like R) is an infinite series of positive powers of p and x, and so for each separate factor.

On multiplying the product of these infinite series by $p\,(1 + Pp)$, we shall necessarily obtain a finite series of the form $p\,(1 + Gp)$. Consequently, the cyclotomic function will divide by p but not by p^2. And we might have used this method exclusively to have established the fact of the first power of p, under the conditions presupposed, being a divisor of the function. This method serves also to establish directly that *every* root of $\chi_{k_1}x \equiv 0$ is a root of the congruence $\chi_k x \equiv 0$ [mod. p]. And we thus see that the intrinsic divisor, when it exists, is a τk_1-fold divisor of the cyclotomic function.

When k is the index to a cyclotomic function, and $k = k_1 p^j$, where p is a prime not contained in k, let us agree to call k_1 the sub-index to p. Then, from what precedes, we may draw the conclusion that a cyclotomic function of the first species has never more than one intrinsic divisor, which, if it exists, is the greatest prime number contained in the index, but is such only in the case when diminished by unity, it contains its own sub-index, (a conclusion necessarily satisfied when the index is a prime, for then its sub-index is unity), and, moreover, that the first power only of such intrinsic divisor, when it exists, is a divisor of the function.

It being true and capable of easy demonstration, that when a rational integer function contains, as a divisor, each of two numbers prime to one

another, their product will also be a divisor of the function, it follows that any number, each of whose prime factors, diminished by unity, contains the index and also every such number multiplied by the highest prime number which is contained in the index (provided that when diminished by unity that prime contains its own sub-index) is a divisor of a cyclotomic function of the first species. This, as I have said, is only another name for that irreducible factor of a binomial x^k-1 whose degree in x is the *totient* of k.

When the cyclotomic function of any species is made homogeneous by the introduction of a second variable y, relatively prime to x, it becomes a form, (in the technical sense of the word), and may then very conveniently be designated a *cyclo-quantic.*

Title 2. *Cyclotomic Functions of the Second Species* (*Conjugate Class*).* I pass on to the theory of the divisors of the function which has for roots the sum of the binomial (*zweigliedrig*) groups of the primitive roots of x^k-1, or, in other words, all the distinct values, $\frac{1}{2}\tau(k)$ in number, of $2\cos\frac{\lambda\tau}{k}$ where λ is any number less than $\frac{k}{2}$ and prime to k.

Such a function, in which the coefficient of the highest power of the variable is supposed to be unity, I call a cyclotomic function, or simply a cyclotomic, of the second species and conjugate class to the index k. It may be found most readily by dividing the corresponding one of the first species, whose variable say is x, by $x^{\frac{1}{2}\tau(k)}$, substituting u for $x+\frac{1}{x}$, and applying for successive values of m the trigonometrical formula for expressing $\cos m\theta$ in terms of powers of $\cos\theta$, except when the index is a prime number, in which case the function in u is given more expeditiously at once by the well-known formula

$$u^m+u^{m-1}-\frac{m-1}{1}u^{m-2}-\frac{m-2}{1}u^{m-3}+\frac{(m-2)(m-3)}{1.2}u^{m-4}+\frac{(m-3)(m-4)}{1.2}u^{m-5}-\ldots,$$

which last coefficient, in the French edition of the Disq. Arith., 1807, it may be worth noting, is written erroneously $\frac{(m-1)(m-4)}{1.2}$.

I have thought it would be useful and convenient for many of my readers to be able to see before them the functions of the two sorts, and I accordingly annex a table of their values for all indices up to 36 inclusive.

* When, in the matter comprehended under this title, by inadvertence, cyclotomic functions of the second species are spoken of without a qualification annexed, it is to be understood, in all cases, that only those of the conjugate class or, in other words, those whose roots are all real, are intended. For brevity I shall usually call this class of functions cyclotomics of the second *sort.*

To the index 1 or 2, the cyclotomic of the second species has no existence. Those of the first species to the index 1 or 2, and of the second to the index 3, 4 or 6 are linear, and of course as forms, have no arithmetical properties, but contain every number as a divisor, linear forms being, as it were, the protoplasm out of which the higher forms are organized.

Table of Cyclotomic Functions of the first species and the conjugate class of the second species for all values of the index from 1 *to* 36 *inclusive.*

Index.	1st Species.	2d Species, Conjugate Class.
1	$x-1$	
2	$x+1$	
3	x^2+x+1	$u+1$
4	x^2+1	u
5	$x^4+x^3+x^2+x+1$	u^2+u-1
6	x^2-x+1	$u-1$
7	$x^6+x^5+x^4+x^3+x^2+x+1$	u^3+u^2-2u-1
8	x^4+1	u^2-2
9	x^6+x^3+1	u^3-3u-1
10	$x^4-x^3+x^2-x+1$	u^2-u+1
11	$x^{10}+x^9+\ldots+x+1$	$u^5+u^4-4u^3-3u^2+3u+1$
12	x^4-x^2+1	u^2-3
13	$x^{12}+x^{11}+\ldots+x+1$	$u^6+u^5-5u^4-4u^3+6u^2+3u-1$
14	$x^6-x^5+x^4-x^3+x^2-x+1$	u^3-u^2+2u+1
15	$x^8-x^7+x^5-x^4+x^3-x+1$	$u^4-u^3-4u^2+4u+1$
16	x^8+1	u^4-4u^2+2
17	$x^{16}+x^{15}+\ldots+x+1$	$u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1$
18	x^6-x^3+1	u^3-3u+1
19	$x^{18}+x^{17}+\ldots+x+1$	$u^9+u^8-8u^7-7u^6+21u^5+15u^4$ $+10u^3-10u^2+5u+1$
20	$x^8-x^6+x^4-x^2+1$	u^4-5u^2+5
21	$x^{12}-x^{11}+x^9-x^8+x^6-x^4$ $+x^3-x+1$	$u^6-u^5-6u^4+6u^3+8u^2-8u+1$
22	$x^{10}-x^9+\ldots-x+1$	$u^5-u^4-4u^3+3u^2-3u+1$
23	$x^{22}+x^{21}+\ldots+x+1$	$u^{11}+u^{10}-10u^9-9u^8+36u^2+28u^6-56u^5$ $-35u^4+35u^3+15u^2-6u-1$
24	x^8-x^4+1	u^4-4u^2+1
25	$x^{20}+x^{15}+x^{10}+x^5+1$	$u^{10}-10u^8+35u^6+u^5-50u^4-5u^3+25u^2-5u-1$

Index.	1st Species.	2d Species, Conjugate Class.
26	$x^{12}-x^{11}+\ldots-x+1$	$u^6-u^5-5u^4+4u^3+6u^2-3u-1$
27	$x^{18}-x^9+1$	$u^9-9u^7+27u^5-30u^3+9u-1$
28	$x^{12}-x^{10}+x^8-x^6+x^4-x^2+1$	$u^6-7u^4+14u^2-7$
29	$x^{28}+x^{27}+\ldots+x+1$	$u^{14}+u^{13}-13u^{12}-12u^{11}+66u^{10}+55u^9-165u^8$ $-120u^7+210u^6+126u^5-126u^4-56u^3+28u^3$ $+7u-1$
30	$x^{16}-x^{14}+x^{10}-x^8+x^6-x^2+1$	$u^8-9u^6+26u^4-26u^2+1$
31	$x^{30}+x^{29}+\ldots+x+1$	$u^{15}+u^{14}-14u^{13}-13u^{12}+78u^{11}+66u^{10}-220u^9$ $-165u^8+330u^7+210u^6-252u^5-126u^4+84u^3$ $+28u^2-4u-1.$
32	$x^{16}+1$	$u^8-8u^6+20u^4-16u^2+2$
33	$x^{20}-x^{19}+x^{17}-x^{16}+x^{14}-x^{13}$ $+x^{11}-x^{10}+x^9-x^7+x^6-x^4$ $+x^3-x+1$	$u^{10}-u^9-10u^8+10u^7+34u^6-34u^5$ $-43u^4+43u^3+12u^2-12u-1$
34	$x^{16}-x^{15}+x^{14}-\ldots+x^2-x+1$	$u^8-u^7-7u^6+6u^5+15u^4-10u^3-10u^2+4u+1$
35	$x^{24}-x^{23}+x^{19}-x^{18}+x^{17}-x^{16}$ $+x^{14}-x^{13}+x^{12}-x^{11}+x^{10}-x^8$ $+x^7-x^6+x^5-x+1$	$u^{12}-u^{11}-12u^{10}+11u^9+54u^8-43u^7-113u^6$ $+71u^5+110u^4-46u^3-40u^2+8u+1$
36	$x^{12}-x^6+1$	$u^6-6u^4+9u^3-3$

A very good test (or, in most cases, pair of tests) of the correctness of the figures is to write $u=\pm 2$* corresponding to $x=\pm 1$ and see if the values for the same index agree. Our interest will presently be concentrated on the single entry in the right hand column, that which expresses the conjugate class of the second species of cyclotomic to the index 9, but the function for the neighboring case of the index 8 is worthy of arresting the reader's attention for a moment, inasmuch as the general theory of cyclotomic divisors applied to it will be seen to supply an instantaneous proof that all prime numbers of the form $8n\pm 1$, and no other prime numbers have 2 for a quadratic residue.†

It is hardly necessary to observe that, when the index is a prime number, it may be duplicated without affecting the character of either set of functions, the only effect produced thereby being the entirely unimportant one of a change in the sign of the variable.

*And a further double test is given by taking $u=0$, $x=2$, as we ought to find $\chi i = 2^{\frac{1}{2}\tau k}\psi 0$.

†So, under the third Title, it will be found that u^2+2 is a *non-conjugate* cyclotomic of the second species to the index 8, of which, according to the general cyclotomic law, the odd prime divisors are of the form $8m+1$ or $8m+3$.

The formula which I have employed for computing $\cos n\theta$ is that which, beginning with the *highest* power of $\cos\theta$, admits of a uniform scheme of setting down the work, which is not the case when the series is started from the other end. It, and the series used for $\dfrac{\sin\frac{p\theta}{2}}{\sin\frac{\theta}{2}}$, also required for my purposes, may be obtained by a much simpler method than any I have seen given in the text books as follows.

In general, the denominator of $\frac{1}{a_1}-\frac{1}{a_2}-\ldots\frac{1}{a_u}$, say the procumulant $[a_1, a_2, \ldots a_u] = A_0 - A_1 + A_2$ etc., where A_0 is $a_1 . a_2 . \ldots . a_u$, A_1 is the sum of the quotients of A_0 by any pair of consecutive elements $a_i . a_{i+1}$, A_2 of the quotients of A_0 by the product of any two such pairs as $a_i . a_{i+1} . a_j . a_{j+1}$, and so on. If we call the *number* of such quotients in A_i, $D_i n$, it is obvious that

$$D_{i+1}n = \Sigma_{t=0}^{t=n-2} D_i t .$$

Hence $D_0 n = 1$, $D_1 n = n-1$, $D_2 n = (n-2)\frac{n-3}{2}$, $D_3 n = \frac{(n-3)(n-4)(n-5)}{1.2.3}$, and so on.

On making $a_1 = a_2 = \ldots = a_n = 2\cos\theta$, it will immediately be seen that the procumulant $[2\cos\theta,\ 2\cos\theta \ldots \text{to } n \text{ terms}]$ expresses $\frac{\sin(n+1)\theta}{\sin\theta}$, because, calling this u_n, the equation in difference for finding it is

$$u_{n+1} = 2\cos u_n - u_{n-1} \text{ and } u_0 = 1 .$$

Consequently $\frac{\sin(n+1)\theta}{\sin\theta} = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{(n-1)(n-2)}{2}(2\cos\theta)^{n-4} \ldots$

Hence $2\cos n\theta = 2\left(\frac{\sin(n+1)\theta}{\sin\theta} - \cos\theta\frac{\sin n\theta}{\sin\theta}\right) = (2\cos\theta)^n - n(2\cos\theta)^{n-2}$ $+ n\frac{n-3}{2}(2\cos\theta)^{n-4} \ldots$ Also, $\dfrac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}} = \frac{\sin(n+1)\theta}{\sin\theta} + \frac{\sin n\theta}{\sin\theta} = (2\cos\theta^n)$ $+ (2\cos\theta)^{n-1} - n(2\cos\theta)^{n-2} - (n-1)(2\cos\theta)^{n-4} + \ldots$ *

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*This expansion Gauss (Rech. Arith., Paris, 1757, p. 431) suggests deriving by means of the exceedingly awkward and unmanageable process indicated by the formula $\frac{\sqrt{1-\cos n\theta}}{1-\cos\theta}$, $\cos n\theta$ being previously supposed to be expanded in terms of powers of $\cos\theta$. *Quandoque bonus dormitat Homerus.*

Writing u in place of $2\cos\theta$ these are the two expansions which I have used to express $x^n + \frac{1}{x^n}$ and $\frac{x^{\frac{p-1}{2}} - x^{-\frac{p-1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}$ in terms of powers of $x + \frac{1}{x}$ in calculating the cyclotomics of the 2d sort whose values are given in the preceding table.

Since $(x^{p-1} - 1)(x^{p+1} - 1) = x^{2p} - x^{p+1} - x^{p-1} + 1$, if, for convenience, we write $x + \frac{1}{x} = u = 2\cos\theta$, it is evident that $\cos p\theta - \cos\theta$, regarded as an algebraical function of $\cos\theta$, will contain all the cyclotomic functions of the second species (conjugate class) whose indices are divisors of $p-1$ or $p+1$ and in addition to these $\left(x - \frac{1}{x}\right)^2$ or $u^2 - 4$ derived from the factor $x^2 - 1$ which is common to $x^{p-1} - 1$ and $x^{p+1} - 1$, but does not give rise to a cyclotomic of this sort until it is squared; $\cos p\theta - \cos\theta$ is thus a product exclusively of cyclotomics of the second sort.

It is well known that $\cos p\theta - \cos\theta \equiv 0$ (mod. p) regarded as a congruence in $\cos\theta$ has the p roots $\cos\theta = 0, 1, 2, 3, \ldots \overline{p-1}$, p being supposed to be a prime number.

But more generally the congruence $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ mod. p^j has its full complement of p^j real roots a theorem, this, which is the analogue of the theorem of Fermat extended to powers of prime numbers put under the form of affirming that $x^{p^j} - x^{p^{j-1}} \equiv 0$ mod. p^j has its full complement of real roots; but, as I do not recall seeing the *cosine* theorem for modulus p^j anywhere stated, and as it is wanted for the theory I am developing, and its truth is not obvious, I shall proceed to prove it. For greater simplicity of notation let us begin with the case where $j = 2$. We have then $\cos p^2\theta = (\cos\theta)^{p^2}$ $- p^2\frac{p^2-1}{2}(\cos\theta)^{p^2-2}.(\sin\theta)^2 + \frac{p^2(p^2-1)(p^2-2)(p^2-3)}{1.2.3.4}(\cos\theta)^{p^2-4}.(\sin\theta)^4\ldots$ and

$$\cos p\theta = (\cos\theta)^p - p\frac{p-1}{2}(\cos\theta)^{p-2}.(\sin\theta)^2 + \frac{p.(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}.(\sin\theta)^4\ldots$$

where of course all the powers of $(\sin\theta)^2$ are regarded as functions of $\cos\theta$. It will easily be recognized that every coefficient in the first series will be divisible by p^2 with the exception of those terms in which a new multiple of p first makes its appearance among the factors of the denominator, which will lose one power of p; the next coefficient to any such as last named taking up a new factor of p into the numerator, the fraction to which it belongs will recover the lost p and be again divisible by p^2.

The difference, therefore, between the two series *quâ mod.* p^2 will be $(\cos\theta)^{p^2} - (\cos\theta)^p$

$$+\frac{p^2(p^2-1)\dots(p^2-2p+1)}{1.2\dots 2p}(\cos\theta)^{p^2-2p}.(\sin\theta)^{2p} - p\frac{p-1}{2}(\cos\theta)^{p-2}(\sin\theta)^2$$

$$+\frac{p^2(p^2-1)\dots(p^2-4p+1)}{1.2\dots 4p}(\cos\theta)^{p^2-4p}.(\sin\theta)^{4p} - \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}.(\sin\theta)^4$$

. . . .

It may be shown that every pair of terms in the above is divisible by p^2 for all real values of $\cos\theta$.

1°. $(\cos\theta)^{p^2} - (\cos\theta)^p$ contains p^2 by Fermat's extended theorem.

2°. *Quâ* p, $(\cos\theta)^{p^2-2p} \equiv (\cos\theta)^{p-2}$ and $(\sin\theta)^{2p} \equiv (\sin\theta)^2$.

Hence *quâ* p^2, the sum of the second pair of terms

$$\equiv p\frac{p-1}{2}\left\{\frac{(p+1)(p-2)(p-3)\dots(p^2-2p+1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0$$

$$\equiv p\frac{p-1}{2}\left\{\frac{2.3\dots(2p-1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0.$$

3°. *Quâ* p, inasmuch as $p^2-5p+4 = (p-1)(p-4)$, $(\cos\theta)^{p^2-4p} \equiv (\cos\theta)^{p-4}$ and $(\sin\theta)^{4p} \equiv (\sin\theta)^4$. Also, $p^n - 1 \equiv p-1$, $p^2-2 \equiv p-2$ and $p^2-3 \equiv p-3$. Hence the sum of the 3d pair of terms *quâ* p^2

$$\equiv \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\left\{\frac{(p^2-4)(p^2-5)\dots(p^2-4p+1)}{4.5\dots(4p-1)}\right\} \equiv 0.$$

And so each pair of terms may be proved to be congruous to zero *quâ* p^2.

The same form of demonstration may be shown to apply to the case of the modulus p^j,* and we may regard as proved the important theorem that $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ [mod. p^j] contains the maximum number of roots p. It follows that $\cos p\theta - \cos\theta \equiv$ mod. p^j will contain p distinct roots. For, if we make $\theta = p^{j-1}\phi$, the congruence becomes $\cos p^j\phi - \cos p^{j-1}\phi \equiv 0$ mod. p^j, which has p^j roots. These roots will separate into p groups of p^{j-1} each, such $\cos(p^{j-1}\phi)$ will be the same for all the $(\cos\phi)$'s in the same group, but different (*quâ* mod. p^j) for any two belonging to distinct groups. For if $\cos\phi_1$ be one of the values regarded as given, and $\cos(p^{j-1}\phi_2) \equiv \cos(p^{j-1}\phi_1)$ mod. p^j,

and $$\left.\begin{array}{l}\cos(p^{j-1}\phi_2) \equiv \cos\phi_2 \\ \cos(p^{j-1}\phi_1) \equiv \cos\phi_1\end{array}\right\}\text{mod. } p.$$

* The reader will please bear in mind that in the expansion of $(a+b)^{p^j}$ the number of coefficients in which p enters to the power $j, j-1, \dots 2, 1, 0$ respectively is $p^j - p^{j-1}, p^{j-1} - p^{j-2}, \dots p^2 - p, p-1, 2$.

If, then, we form the series

$$\cos\phi_1,\ \cos\phi_1+p,\ \cos\phi_1+2p,\ \ldots\ \cos\phi_1+(p^{j-1}-1)\,p,$$

all the values of $\cos\phi_2$ must be included among the terms of this series. Conversely, if we make $\cos\phi_2=\cos\phi_1+\lambda p$, we shall have

$$\cos p^{j-1}\phi_2-\cos p^{j-1}\equiv 0 \text{ mod. } p^j.$$

For, writing q for p^{j-1},

$$\cos q\phi_2=(\cos\phi_2)^q-q\,\frac{q-1}{2}\,(\cos\phi_2)^{q-2}.\,(\sin\phi_2)^2+\ldots.$$

If in this development we take the term containing $(\cos\phi_2)^{q-2t}.\,(\sin\phi_2)^{2t}$, its coefficient will contain q, except in the case where t contains p^i, in which case the coefficient will contain $\frac{q}{p^i}$ but not q, and the index of $(\cos\phi_2)$ and $(\sin\phi_2)^2$ will each contain p^i. Hence, since $\cos\phi_2=\cos\phi_1+\lambda p$, and consequently $(\sin\phi_2)^2$ is of the form $(\sin\phi_1)^2+\Delta p$, it follows that the difference between this term and the corresponding one in the development of $\cos q\phi_1$ will in the one case contain qp and in the other $\frac{q}{p^i}\,p^{i+1}$, in either case therefore it contains $p\,.\,q$, *i. e.* p^j, and consequently making $\cos\phi_2$ equal to any of the p^{j-1} terms of the series, we shall have $\cos\,(p^{j-1}\phi_2)\equiv\cos\,(p^{j-1}\phi_1)$ mod. p^j as was to be shown. Hence $\cos p\theta-\cos\theta\equiv 0$ mod. p^j will have p real roots.

Again no algebraical factor of $\cos p\theta-\cos\theta$ can have a *superfluity* of real roots *quâ* mod. p^j, for if it had then by the same reasoning as applied to the cyclotomics of the first species, it would be necessary for p to be contained in the discriminant of $\cos p\theta-\cos\theta$ regarded as a function of $\cos\theta$, but *quâ* mod. p, this is the same as the discriminant of $(\cos\theta)^p-\cos\theta$ in regard to $\cos\theta$ or of x^p-x in regard to x which is the discriminant of $x^{p-1}-1$ multiplied by the squared resultant of x and $x^{p-1}-1$, and is therefore a power of $(p-1)$. Hence every algebraical factor of $\cos p\theta-\cos\theta$ *quâ mod.* p^j contains *its full quota* of real roots, *i, e.* as many roots as there are units in its degree.

If then $p=mk+\varepsilon$, where $\varepsilon=\pm 1$, since $\cos p\theta-\cos\theta$ will contain the cyclotomic of the second sort to the index k, such cyclotomic equivalented to zero [mod. p^j] will have all its roots real, so that $(mk\pm 1)^j$ will be a $\frac{\tau k}{2}$-fold divisor of such function.

As in the case of cyclotomics of the 1st species we may separate the divisors of those of the 2d sort into intrinsic and extrinsic, according as they are or are not divisors of the index.

First, as regards the extrinsic divisors, we may prove that no other prime numbers except those of the form $k \pm 1$ can be divisors of the 2d species of cyclotomics to the index k.

To show this I proceed as follows: $\psi_k u$ is contained algebraically in $\dfrac{\sin \frac{k}{2}\theta}{\sin \frac{\theta}{2}}$, and *à fortiori* in its square, *i. e.* in $\dfrac{1-\cos k\theta}{1-\cos\theta}$, so that if $2\cos\theta$ is a value of u, which makes $\psi_k u$ contain p,

$$\cos k\theta \equiv 1 \text{ mod. } p\,,$$

but also $\cos p\theta \equiv \cos\theta$ mod. p, and if $\dfrac{\sin p\theta}{\sin\theta} \equiv a + bp$,

$$1 = (\cos\theta)^2 + a^2(1-\cos\theta)^2 + cp\,,$$

and $(1-a^2)(1-\cos\theta)^2 = cp$, and, therefore, $a = \pm 1$ mod. p, for $\dfrac{1-\cos k\theta}{1-\cos\theta}$ does not contain $(1-\cos\theta)$, and if $(1-\cos k\theta)$ contains $1-(\cos\theta)^2$, which is only the case when k is even, $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, does not contain either $1-\cos\theta$ or $1+\cos\theta$, and, therefore, $\psi_k u$, which, in that case, is contained in $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, will not contain either $1-\cos\theta$ or $1+\cos\theta$.

Hence $1-(\cos\theta)^2$ is not zero, and, consequently, $a \equiv \pm 1$, and, therefore, $\dfrac{\sin p\theta}{\sin\theta} \equiv \pm 1$ mod. p.

Hence, either

$$\left.\begin{array}{l} \cos(p-1)\theta = \cos p\theta \,.\, \cos + \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \\ \text{or} \\ \cos(p+1)\theta = \cos p\theta \,.\, \cos\theta - \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \end{array}\right\} \text{mod. } p,$$

and writing $\varepsilon = \pm 1$, we must have

$$\cos(p-\varepsilon)\theta \equiv 1 \text{ mod. } p\,.$$

If possible, let $(p-\varepsilon)$ not contain k, and δ (less than k) be the greatest common measure of k and $(p-\varepsilon)$.

Let $\lambda(p-\varepsilon) - \mu k = \delta$. Then

$$\left.\begin{array}{ll} \cos\lambda(p-\varepsilon)\theta \equiv 1 & \dfrac{\sin\lambda(p-\varepsilon)\theta}{\sin\theta} \equiv 0 \\ \cos\mu k\theta \equiv 1 & \dfrac{\sin pk\theta}{\sin\theta} \equiv 0 \end{array}\right\} \text{mod. } p\,.$$

Hence $\cos\delta\theta \equiv 1$ mod. p, and, consequently, the resultant of $\psi_k u$ and $\cos\delta\theta - 1$ in respect to $\cos\theta$ must contain p. But $\psi_k u$, when δ is any divisor of k other than k itself, is an algebraical factor of $\frac{\cos k\theta - 1}{\cos\delta\theta - 1}$ *à fortiori*, therefore, the resultant of this last named function of $\cos\theta$ and of $\cos\delta\theta - 1$ must contain p.

This resultant will be the product of the values of $\frac{\cos k\theta - 1}{\cos\delta\theta - 1}$ for every root of $\cos\delta\theta - 1$, it is therefore the δth power of the value of the vanishing fraction $\frac{\cos\mu\varphi - 1}{\cos\varphi - 1}\left[\text{where } \mu = \frac{k}{\delta}\right]$ when $\cos\phi = 1$, *i. e.* of $\left(\frac{\sin\frac{\mu}{2}\varphi}{\sin\frac{\varphi}{2}}\right)^2$ when $\phi = 0$. The resultant is, therefore, $\left(\frac{k}{\delta}\right)^{2\delta}$, which cannot contain p, since, by hypothesis, p is not contained in k. Hence $p - \varepsilon = mk$, or $p = mk \pm 1$. So that, for the extrinsic divisors, the law, both as regards what numbers are and what are not such divisors, is precisely the same as for the cyclotomics of the first species, except that $mk \pm 1$ takes the place of $mk + 1$.

Next, for the intrinsic divisors. Suppose p to be any such, and that $k = k_1 p^j$, where k_1 is prime to p. Then p is a divisor of $\cos k_1(p^j\theta) - 1$, and, therefore, by what has been shown, must be of the form $mk_1 \pm 1$, unless $(\cos p^j\theta - 1)$ contains p, in which case, since

$$\cos p^j\theta = (\cos p^j\theta - \cos p^{j-1}\theta) + (\cos p^{j-1}\theta - \cos p^{j-2}\theta) + \ldots + \cos\theta,$$

$\cos\theta - 1$ must contain p, and, consequently, p must be a divisor of $\psi_k 2$, *i. e.* of $\chi_k 1$, which we have seen is equal to 1, except when $k_1 = 1$. Hence, p must be of the form $mk_1 \pm 1$. To show the converse, that when $k = k_1 p^j$ and $p = mk_1 \pm 1$, p will be a divisor of $\psi_k u$. Taking, first, the case of $k_1 = 1$ or $k = p^j$, $\psi_k u$, for $u = 2$ will be equal to $\chi_k 1$, which, as we have seen, will divide by p, and not by p^2.

To ascertain if there is any other value of u which will make the function divisible by p, I observe that, for this case, $(\psi_k u)^2 = \frac{\cos p^j\theta - 1}{\cos p^{j-1}\theta - 1}$, which is of the form $\frac{\cos\theta - 1 + Lp}{\cos\theta - 1 + lp}$, and if this function contains p, we must obviously have $\cos\theta \equiv 1$ mod. p.

Proceeding to the more general case where $k = k_1 p^j$ and k_1 is other than unity, taking as I did for the first species the specimen case $k = k_1 p$, $k_1 = abc$,

$p = mk_1 \pm 1$, we shall have

$$(\psi_k u) = \frac{(\cos abcp\theta - 1)(\cos ab\theta - 1)(\cos ac\theta - 1)(\cos bc\theta - 1)(\cos ap\theta - 1)(\cos bp\theta - 1)(\cos cp\theta - 1)(\cos\theta - 1)}{(\cos abc\theta - 1)(\cos abp\theta - 1)(\cos acp\theta - 1)(\cos bcp\theta - 1)(\cos a\theta - 1)(\cos b\theta - 1)(\cos c\theta - 1)}.$$

If, now, $\cos k_1\theta - 1 = 0$, and we suppose $\cos\theta$ to be a root of $\psi_k u = 0$, $\cos p\theta = \cos(\pm\theta) = \cos\theta$, $(\psi_k u)^2$ becomes equal to $\frac{\cos pk_1\theta - 1}{\cos k_1\theta - 1} = p$, and paying no attention to the algebraical sign which is immaterial to our object, we shall have $\psi_k u = p$, and the resultant of $\psi_{k_1} u$ and $\chi_k u$ will be $p^{\frac{1}{2}\tau k_1}$, and, consequently, since $\chi^{k_1} u \equiv 0$ mod. p has all its roots real, one of them, at all events, will belong to $\chi_k u \equiv 0$ mod. p, and precisely in like manner, as in the case for cyclotomics of the 1st species, it may be shown that this reasoning ceases to apply if $\cos\theta$, although satisfying $\cos k_1\theta - 1 = 0$, does not satisfy $\chi^{k_1} u = 0$, in which case the resultant, instead of being a power of p, would become unity, so that the value of $\cos\theta$, satisfying $\cos k_1\theta - 1 \equiv 0$ mod. p, could not be a congruence root of $\chi_k u \equiv 0$ mod. p. Finally, as for the case of the 1st species, it may be shown that *every* congruence root of $\chi^{k_1} u \equiv 0$ [when $k = k_1 p^j$ and $p = mk_1 \pm 1$] will satisfy the congruence $\chi_k u \equiv 0$ mod. p, and that only p, and not p^2, will be a divisor of $\chi_k u$, subject, however, to an exception for the case of $p = 2$, when $k = 2$ or $k = 4$, and also for the case of $p = 2$ and $p = 3$ when $k = 6$.* As regards these intrinsic divisors, it is clear that any root must be the highest prime factor of the index unless its sub-index is 3, in which case it may be 2. It is obvious, then, that except the index is 6 or 12, the second cyclotomic function can have only one intrinsic divisor. When the index is 6, the function is simply $u - 1$, and contains of course *every power* of 2 and 3, as well as every power of $6i \pm 1$ as a divisor.

Leaving out of consideration the three known cyclotomics, whose indices are 3, 4, 6, and the one just referred to, $u^2 - 3$, whose index is 12, we may combine the results obtained into the statement that any number, each of whose factors, diminished or increased by unity, contains the index, and any such number, multiplied by the highest prime number in the index, provided that that number, when increased or diminished by unity, contains its sub-index, and no other numbers but such as satisfies one or the other of these two descriptions, will be a divisor of a non-linear cyclotomic function of the conjugate class of the second species whose index is other than 12. As regards

*I may probably show this in full in a future note. But since the vast and dazzling theory for cyclotomics of all species, with an indefinite number of classes to each species, has loomed into view, I must confess to a certain feeling of impatience as regards working out these small details for a single class of a single species. The inordinately augmented amplitude of the subject calls for some broader method of treatment.

the index 12, any number, whose factors are all of the form $12m \pm 1$, as also the double, treble and sextuple of any such number, will be a divisor of the function.

By way of example let us consider the indices 15, 21, 35.

$\chi_{15}x$ will contain neither 3 nor 5, $\psi_{15}x$ will contain 5 but not 3.

$\chi_{21}x$ will contain 7 but not 3, $\psi_{21}x$ will contain 7 but not 3.

$\chi_{35}x$ will contain neither 5 nor 7, $\psi_{35}x$ will contain neither 5 nor 7.

To find a value of x which make $\psi_{15}x$ contain 5, write $\psi_3 u = u + 1 \equiv 0$ mod. 5, then $u \equiv -1$.

To find values of x which make $\psi_{21}x$ contain 7, write $u + 1 \equiv 0$ mod. 7, then $u \equiv 6$; and to find values of x which make $\chi_{21}x$ contain 7, write $x^2 + x + 1 \equiv 0$ mod. 7, then $x \equiv 2$ or $x \equiv 4$.

On turning to the table p. 367 it will be seen that

$$\psi_{15}(-1) = 1 + 1 - 4 - 4 + 1 = -5,$$

$$\psi_{21}(-1) = 1 + 1 - 6 - 6 + 8 + 8 + 1 = 7,$$

$$\left.\begin{aligned}\psi_{21}2 = 4096 + 512 + 64 + 8 + 1\\ - 2048 - 256 - 16 - 2\end{aligned}\right\} = 4681 - 2322 = 2359 = 7\,.\,(16\,.\,21 + 1),$$

and of course since $\chi_{21}x^2$ contains $\chi_{21}x$ as an algebraical factor, $\chi_{21}4$ will also contain the intrinsic divisor 7 on the general principle that if λ be any number prime to k, $\chi_k x^\lambda$ must contain $\chi_k x$ as an algebraical factor, as admits of easy demonstration.

Also $\psi_{21}6 \equiv \psi_{21}\left(2 + \frac{1}{2}\right) \equiv \chi_{21}2$ [mod. 7] will also contain 7. Lastly, to mod. 5, for $x = 0, 1, 2, 3, 4$

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 1, 1, 1, 1;$$

and to mod. 7, for $x = 0, 1, 2, 3, 4, 5, 6,$

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 2, 1, 3, 3, 1, 2;$$

so that neither 5 nor 7 is a divisor of either function to index 35.

Title 3. *On Cyclotomic Functions of Any Species and Class.* The cyclotomic functions, called by me, of the second *sort* or conjugate class of the second species discussed under the preceding title, constitute the leading class of a much more general kind of binomial (*zweigliedrig*) cyclotomics, which it would mislead were I to suppress all allusion to.

Suppose k to contain θ distinct odd prime factors, then we know that the number of square roots of unity to the modulus k is 2^θ, except when k is divisible by 4, in which case it is $2^{\theta+1}$, or $2^{\theta+2}$, according as $\frac{k}{8}$ is fractional or integer, or, setting apart unity, the number remaining is $2^\theta - 1$,

$2^{\theta+1}-1$, $2^{\theta+2}-1$ in the three cases respectively. Let $\sqrt{1}$ (one of the totitives to k) denote any *specific one* of these square roots. Then, if we call ρ any primary kth root of unity and make $x=\rho+\rho^{\sqrt{1}}$, we shall obtain a completely integer function of the degree $\frac{1}{2}\tau k$ in x, which may be called a binomial cyclotomic. When k is divisible by 4, one value of $\sqrt{1}$ will be $\frac{k}{2}+1$, and the value of $\rho+\rho^{1+\frac{k}{2}}$ being zero, the cyclotomic function that ought to be, degenerates into a power of x. Hence, when k is not divisible by 4, the number of binomial cyclotomics is $2^{\theta}-1$, when it is divisible by 4, $2^{\theta+1}-2$, or the double of the former value, and when by 8, $2^{\theta+2}-2$.

All these binomial cyclotomics will be found to possess similar properties to those which have been demonstrated under Title 2 concerning their leading class, as the annexed examples will serve to demonstrate, where the odd prime extrinsic factors it will be seen are of the form $mk+1$ or $mk+\sqrt{1}$; that is to say, in respect to the index, are congruous to one or the other of the *primordinal* totitives 1 and $\sqrt{1}$ where the latter quantity has a definite value for each of the cyclotomics in question.

Thus, suppose $k=15$, the square roots of unity (*quâ* 15) are ± 1, ± 4. Let $\sqrt{1}=4$, and make $x=\rho+\rho^4$, then it will be found that $x^4-x^3+2x^2+x+1$ will contain the four roots of x and all the odd prime divisors of this function are of the form $15m+1$, 4.

Or, again, let $\alpha=\rho+\rho^{11}$, then it will be found that x is a root of the function $x^4+x^3+x^2+x+1$, the prime factors of which, other than 5, are of the form $15m+1$, 11, which is, in effect, the same as the form $5m+1$.

Again, let $k=20$. The values of $\sqrt{1}$ [mod. 20] are ± 1, ± 9. If we were to put $x=\rho+\rho^{11}$, its value would be zero, but writing $x=\rho+\rho^9$, we shall find it will be the root of x^4+3x^2+1, all the prime factors of which, other than the intrinsic one 5, are of the form $20m+1$, 9.*

We may now proceed to generalize these results by considering cyclotomics of every possible numerosity of grouping for a given index, and of every possible order of conjunction for a given numerosity—in a word, we are brought face to face with the most general theory of ν-nomial cyclotomic functions.†

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* If $k=8$ and we take $x=\rho+\rho^3$ it will be a root of x^2+2 of which the odd extrinsic factors will be of the form $8m+1$, 3.

† All the species with their several classes here referred to form but a single genus of cyclotomic functions. The second genus will arise from the subdivision of groups into smaller groups and so on continually.

I have accordingly calculated cyclotomic functions for the cases following:

$k=15$	$\mu=2$	$\nu=4$
$k=21$	$\mu=4$	$\nu=3$
	$\mu=3$	$\nu=4$
	$\mu=2$	$\nu=6$
$k=26$	$\mu=4$	$\nu=3$
	$\mu=2$	$\nu=6$
$k=28$	$\mu=4$	$\nu=12$
	$\mu=2$	$\nu=6$
$k=25$	$\mu=5$	$\nu=4$
$k=33$	$\mu=5$	$\nu=4$
	$\mu=4$	$\nu=5$
	$\mu=2$	$\nu=10$

Understanding by the "totitives" of k the numbers less than k and prime to it, these totitives may be arranged in (among others) the natural groups hereunder written.

Totitives to 15 for $\mu=2$, $\nu=4$

1	4	11	14
2	7	8	13

" to 21 for $\mu=4$, $\nu=3$

1	4	16
2	8	11
5	17	20
10	13	19

" " for $\mu=3$, $\nu=4$

1	8	13	20
2	5	16	19
4	10	11	17

" " for $\mu=2$, $\nu=6$

1	4	5	16	17	20
2	8	10	11	13	19

" to 26 for $\mu=4$, $\nu=3$

1	3	9
5	15	19
7	11	21
17	23	25

Totitives to 26 for $\mu = 3$, $\nu = 4$

1	5	21	25
3	11	15	23
7	9	17	19

" " 28 for $\mu = 4$, $\nu = 3$

1	9	25
3	27	19
5	17	13
11	15	23

" " for $\mu = 2$, $\nu = 6$

1	3	9	19	25	27
8	10	11	17	18	23

" to 25 for $\mu = 5$, $\nu = 4$

1	7	18	24
2	11	14	23
3	4	21	22
6	8	17	19
9	12	13	16

To save space, I omit the groupings to $k = 33$.

If, in any of the above tables, we call the totitives of the several rows,

$$\begin{array}{llll} \tau_{1,1} & \tau_{1,2} & \dots & \tau_{1,\nu} \\ \tau_{2,1} & \tau_{2,2} & \dots & \tau_{2,\nu} \\ \dots & \dots & \dots & \dots \\ \tau_{\mu,1} & \tau_{\mu,2} & \dots & \tau_{\mu,\nu} \end{array}$$

and if ρ be a primitive root of $x^k - 1$, and we write $R_\theta = \rho^{\tau_{\theta,1}} + \rho^{\tau_{\theta,2}} + \dots \rho^{\tau_{\theta,\nu}}$, $R_1, R_2, \dots R_\mu$ will be the roots of a cyclotomic of the νth species to the index k, or, as we may say, the index k and *nome* ν.

The values of the cyclotomic functions may be found most easily by calculating all the values of σ_i (the sum of the ith powers of its roots), from $i = 1$ to $i = \mu$ where $\mu = \frac{\tau(k)}{\nu}$.

The value of $X_{k,\nu}$ will then be the sum of the terms not containing negative powers of x in the development of $x^\mu \left\{ e^{-\frac{\sigma_1}{x} - \frac{\sigma_2}{2x^2} - \dots - \frac{\sigma_\mu}{\mu x^\mu}} \right\}$.

It will, of course, be recognized that the first row of numbers (the primordinal totitives, as we may term them) in any of the foregoing natural

schemes of decomposition of the kth primitive roots of unity into groups are νth roots (not necessarily comprising any primitive root) of unity in respect to the index k as modulus.

The values of the cyclotomics are exhibited in the annexed table.

Index.	Nome.	Cyclotomic function.	Primordinal Totitives.
15	4	$x^2 - x - 1$	1, 4, 11, 14
21	3	$x^4 - x^3 - x^2 - 2x + 4$	1, 4, 6
"	4	$x^3 - x^2 - 2x + 1$	1, 8, 13, 20
"	6	$x^2 - x - 5$	1, 4, 5, 16, 17, 20
26	3	$x^4 - x^3 + 2x^2 + 4x + 3$	1, 3, 9
"	4	$x^3 - x^2 - 4x - 1$	1, 5, 21, 25
28	3	$x^4 - 3x^2 + 4$	1, 9, 25
"	6	$x^2 - 7$	1, 3, 9, 19, 25, 27
25	4	$x^5 - 10x^3 + 5x^2 + 10x + 1$	1, 7, 18, 24
33	4	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$*	$\pm 1, \pm 10$
"	5	$x^4 - x^3 - 2x^2 - 3x + 9$	$1, -2, 4, -8, 16$
"	10	$x^2 - x - 8$	$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

In each of the above cases calling the index k, its totient $\mu\nu$, the nome ν and the primordinal totitives $\theta_1, \theta_2 \ldots \theta_\nu$, it will be found that all the *odd* extrinsic prime number divisors (*i. e.* primes dividing the function but not its index) are of the form $mk + \theta_1, \theta_2, \ldots \theta_\nu$.

Here, for the present, I must be content to leave this great theory, or I should be in danger of never finding my way back from it to the original object of the memoir which, although its parent, it transcends in importance; for Bachmann's work, as it seems to me, gives proof, that Cyclotomy is to be regarded not as an incidental application, but as the natural and inherent centre and core of the arithmetic of the future.

Remark on the intrinsic divisors of cyclotomic functions of the 1st species.

It has been seen that if $k = \frac{p-1}{m} p^{j-1} = k_1 p^{j-1}$, $\chi_k x \equiv 0$ *mod.* p^j has all its roots the same as those of $\chi_{k_1} x \equiv 0$ mod. p and does not contain p^2. If, then, we make j successively $0, 1, 2 \ldots j-1$ it will follow that $\chi_{k_1}, \chi_{k_1 p}, \chi_{k_1 p^2}, \ldots \chi_{k_1 p^{j-1}}$ will each contain p, but only in the first power for the same τk_1 values of x. Hence $x^{\frac{(p-1)p^{j-1}}{m}} - 1$, which contains all the above written cyclotomics, will

* The values of $\sigma_2, \sigma_3, \sigma_4, \sigma_5$ in this case follow the noticeable progression 9, 4, 25, 16.

contain p^j, so that $x^{\frac{\tau p^j}{m}} - 1 \equiv 0$ mod. p^j will have $\tau\left(\frac{p-1}{m}\right)$ primitive roots, and it is easy to see that $x^{\frac{k}{n}} - 1$ will not have any congruence root in common with $x^{k_1} - 1$ in respect to the modulus p^j.

The theory of intrinsic divisors, it will thus be seen, contains within itself the whole theory of primitive roots, which I notice because it induces me to withdraw the remark made in a previous foot note that the exact determination of the properties of the intrinsic cyclotomic divisors is a matter of comparatively small importance.

NOTES TO PROEM.

1. *On the rational in- and- exscribed triangle to the cubic curve* $x^3 - 3xy^2 - y^3 + 3z^3 = 0$.

IN the proem it was, under another form of expression, intimated in advance of what will be shown in the second section of this chapter, that the curve $x^3 + y^3 + Az^3 = 0$ has a correspondence with the curve $x^3 - 3xy^2 - y^3 + 3Az^3 = 0$, of such a kind that whenever the second equation has a rational solution, the same must be true of the first, so that (*ex. gr.*) on making $A = 1$, the solubility of $x^3 - 3xy^2 - y^3 + 3z^3 = 0$ in integers implies the like of the equation $x^3 + y^3 + z^3 = 0$. Hence it might, at first sight, be rashly inferred (which is what happened to me when writing the 2d foot-note to page 284 from a sick bed) that since a cube number cannot be broken up into the sum of two others, the former of these last written equations is insoluble in integers. But the fact stares one in the face that it has three solutions in integers, viz:

$$x:y:z:: \ 1: \ 1:1$$
$$x:y:z:: -2: \ 1:1$$
$$x:y:z:: \ 1:-2:1$$

In general, (except at points of inflexion or at points whose ith tangentials are points of inflexion*), one rational point in a cubic gives rise to an infinite series of rational derivatives, but in this case the three points $1:1:1$, $-2:1:1$, $1:-2:1$ are the angles of a triangle in- and- exscribed to the curve $x^3 - 3xy^2 - y^3 + 3z^3$, and are the only rational points on the curve. Each of them is its own third tangential, so that, at any one of the three, an

* Thus we have the following distinction of cases as regards the algebraically rational derivatives of any point on a cubic curve: 1°. An infinite succession of links. 2°. A finite open chain reducing in the case of inflexions to a single point. 3°. A closed chain with a finite number of links.

infinite number of cubic curves can be made to pass having plethoric, or, so to say, pluperfect contact with each other (9-point contact) and accordingly will not intersect each other in any other point.

To these three points will be found to correspond (as will presently be shown in § 2) points for which x or y is zero in the curve $x^3+y^3+z^3=0$. This perfectly explains the seeming paradox.

The sides of the rational in- and- exscribed triangle are easily seen to be $y-z=0$, $x+y+z=0$, $x-z=0$.

In general, if any cubic be thrown into the form $x^2y+y^2z+z^2x+\lambda xyz$, it will obviously be in- and- exscribed to the triangle x, y, z*. In the present instance, if we write $x-z=u$, $y-z=v$, $x+y+z=-w$, it will be found that the curve $x^3-3xy^2-y^3+3z^3$ becomes simply $uv^2+vw^2+wu^2$, of which the Hessian is the three straight lines $u^3+v^3+w^3-3uvw$. If we take the sides of an equilateral triangle whose area is $\frac{1}{2}\Delta$ for the axes of u, v, w, we shall have $u+v+w=\Delta$, and the three real points of inflexion being in the line $u+v+w$, will pass off to infinity, so that the curve will possess three infinite branches. Writing $\omega=\frac{2\pi}{9}$, each asymptote will cut the sides of the angles of reference in 3 pairs of segments abutting at the several angles, such that the ratio to each other of the segments in the several pairs, taken in regular order, will be (for the three asymptotes respectively),

$$\frac{\cos\omega}{\cos 2\omega},\quad \frac{\cos 2\omega}{\cos 4\omega},\quad \frac{\cos 4\omega}{\cos\omega},$$
$$\frac{\cos 2\omega}{\cos 4\omega},\quad \frac{\cos 4\omega}{\cos\omega},\quad \frac{\cos\omega}{\cos 2\omega},$$
$$\frac{\cos 4\omega}{\cos\omega},\quad \frac{\cos\omega}{\cos 2\omega},\quad \frac{\cos 2\omega}{\cos\omega}.$$

These ratios, of course, remain the same, for the conjugate cubic $u^2v+v^2w+w^2u$, except that the order of the readings has to be reversed.

According to my departed friend, (of cherished memory), Otto Hesse's dictum, I suppose it may almost be taken for granted without proof, which would obviously be easy, that the two sets of real asymptotes for the conjugate cubics will envelop one and the same conic.

In a future excursus I propose to demonstrate the existence of an infinite number of polygons in- and- exscribable about any given cubic, and to deter-

*For x will touch the cubic at x, y; y at y, z; z at z, x.

mine the number of such polygons for any existent number of sides. Since $uv^2 + vu^2 + uw^2 = 0$ is equivalent to $(2uw + v^2)^2 + (4u^3v - v^4) = 0$, we are able to deduce, from the fact that one cube cannot be the sum of two others, the theorem that the equation $v^4 - 4u^3v = t^2$ has no solution in integers,* (zeros excluded) which seems to me (the way in which it is got, I mean, not the theorem itself) a very surprising inference.

SCHOLIUM. *On triangles and polygons in- and- exscribable to a general cubic.*

The apices of any such triangle must be points which are their own 3d tangentials. Any such point, it may be shown, is completely defined by the condition that two right lines, drawn, the first through it and any one chosen at will, of the 9 points of inflexion, the second through its tangential and some other point of inflexion, shall meet the curve in the same point.

If, then, the cubic be written under its canonical form, and we select the point of inflexion (I), for which $x = 1$, $y = 1$, and through the point $P(x, y, z)$, which is to be its own 3d tangential, and I draw a ray meeting the curve in P', and through P' and Q, the tangential to P, [*i. e.* the point whose coordinates are $x(y^3 - z^3)$, $y(z^3 - x^3)$, $z(x^3 - y^3)$] draw a ray, the point (X, Y, Z), where that ray meets the curve, must be a point of inflexion, and, *vice versa*, if the condition is fulfilled, P is its own 3d tangential.

It will be found that

$$
\begin{aligned}
&X: -x^6y^3 - y^6z^3 - z^6x^3 + 3x^3y^3z^3 \\
::\ &Y: -x^3y^6 - y^3x^6 - x^3z^6 + 3x^3y^3z^3 \\
::\ &Z: \quad xyz\,(x^6 + y^6 + z^6 - x^3y^3 - y^3z^3 - z^3x^3)\,,
\end{aligned}
$$

and we must have $X = 0$ or $Y = 0$ or $\frac{Z}{xyz} = 0$, the factor which figures in Z being disregarded, because it would lead to the 9 points of inflexion, which

*Suppose the equation $u^2v + v^2w + w^2u = 0$ is resoluble in non-zero integers. We may regard u, v, w as having no common measure, as any such, if it existed, could be driven out of the equation by division. Suppose p to be any prime number entering exactly a times into u and β times into v; then writing $u = p^a u_1$, $v = p^\beta v_1$, since w^2u contains p^a, and v^2w, $\beta^{2\beta}$, we must have $a = 2\beta$ and $p^{3\beta}u_1^2v_1 + v_1^2w + w^2u_1 = 0$, and proceeding similarly with each prime common measure of u, v of v, w and of w, u, it is obvious that, calling the greatest common measure of these three pairs δ, ε, θ, we must have $\delta^3u'^2v' + \varepsilon^3v'^2w' + \theta^3w'^2u' = 0$, where u', v', w' have no two of them any common measure. Hence, apart from algebraical sign u', v', w' must be each of them unity, and the above equation may be written $\delta_1^3 + \varepsilon_1^3 + \theta_1^3 = 0$, the same in form as that which gave birth to the equation $\xi^3 - 3\xi\eta^2 + \eta^3 = 0$, of which $u^2v + v^2w + w^2u = 0$ is a transformation. It is worthy also of remark that the two equations $u^2v + v^2w + w^2u = 0$ and $x^3 + y^3 + z^3 = 0$ pass into one another through the medium of the self-reciprocal substitution-matrix

$$
\begin{matrix}
1 & 1 & 1 \\
\rho^{\frac{1}{3}} & \rho^{\frac{4}{3}} & \rho^{\frac{7}{3}} \\
\rho^{\frac{2}{3}} & \rho^{\frac{8}{3}} & \rho^{\frac{5}{3}},
\end{matrix}
$$

where ρ is a primitive cube root of unity.

may be thrown out of account, as for each of them the in- and- exscribed triangle reduces to a point.

Combining each of the above equations taken separately with the equation to the cubic, we see that there will be $3 \times (9 + 9 + 6)$, *i. e.* 72 points forming the apices of 24 in- and- exscribed triangles to the cubic. It may be shown further that these 24 triangles consist of 12 pairs of conjugate triangles, every pair being so situated that each is a threefold perspective representation of the other, the three perspective centres being some one of the 12 sets of 3 collinear points of inflexion.*

The 24 in- and- exscribed triangles may therefore be distributed into 4 groups, each containing 3 pairs of conjugate triangles. This theory and the general one of in- and- exscribed polygons with any number of sides to a cubic curve will be treated more fully in a future excursus. It may, however, be remarked here that the equation $\frac{Z}{xyz} = 0$ is equivalent to the two $x^3 + \rho y^3 + \rho^2 z^3 = 0$, and $x^3 + \rho^2 y^3 + \rho z^3 = 0$, so that 18 of the points xyz may be found by solving two cubic equations between x^3, y^3 or y^3, z^3 or z^3, x^3. The remaining 54 may be found by substituting for x, y, z respectively (in the simple equations which express their ratios)

$$\begin{array}{llll} 1^\circ. & x + y + z & x + \rho y + \rho^2 z & x + \rho^2 y + \rho z \\ 2^\circ. & x + y + \rho z & x + \rho y + z & \rho x + y + z \\ 3^\circ. & x + y + \rho^2 z & x + \rho^2 y + z & \rho^2 x + y + z \end{array}$$

(these substituted values, together with the original values of x, y, z, rep-

---

**ABC*, *LMN* are in threefold perspective when *AL*, *BM*, *CN*; *AM*, *BN*, *CL*; *AN*, *BL*, *CM* meet in three several points. If *ABC* be taken as the triangle of reference and the coordinates of *L*, *M*, *N* are a, b, c; a', b', c'; a'', b'', c'' respectively, the triple "perspectivische lage" requires only the satisfaction of two conditions, viz: $ab'c'' = bc'a'' = ca'b''$, so that there is nothing between single and triple perspective relation. This statement constitutes a porism. The double condition $ba'c'' = cb'a'' = ac'b''$ of course corresponds to the contrary relation of triple perspective where *AM*, *BL*, *CN*; *AL*, *BN*, *CM*; *AN*, *BM*, *CL* meet in three several points.

Let I, I', I'', J, J', J'', K, K', K'' denote three points of collinear inflexions and P, Q the 3d point collinear with P and Q any two points on the cubic. If Q is the tangential to P, one of the vertices in question, it may be proved that any inflexion I, being assumed, another J may be found such that $IP = JQ$. From this it follows that PQ will satisfy the 10 equations

$$\begin{array}{lll} & PP = Q & \\ IP = JQ & JP = KQ & KP = IQ \\ I'P = J'Q & J'P = K'Q & K'P = I'Q \\ I''P = J''Q & J''P = K''Q & K''P = I''Q. \end{array}$$

These will necessarily continue to be satisfied when I and J are interchanged, provided that $4P$, Q be written KP and KQ or $K'P$ and $K'Q$ or $K''P$ and $K''Q$, and, consequently, to P, Q, R one in- and- exscript, will correspond another denotable indifferently by KP, KQ, KR, $K'P$, $K'Q$, $K'R$, $K''P$, $K''Q$, $K''R$, which will obviously therefore be in triple *perspectivische lage* with the first named one.

resenting the sides of the 4 triangles which contain 3 points of inflexion on each side).*

We may thus neglect altogether the equations $X=0$, $Y=0$, the values of x, y, z, to which they would lead, being comprised among those resulting from the above method.†

In like manner, as we have found the number of in- and- exscribable triangles, it may be shown that the number of quadrilaterals in- and- exscribable to a cubic is 54, and of p-laterals, when p is a prime number, $8(2^{p-1}-1)(2^{p-2}+1)$. For a k-sided polygon, where k is any number whatever, the rule is as follows. Let

$$\phi x = 8\,(2^{x-1}-(\overline{1})^{x-1})(2^{x-2}-(\overline{1})^{x-2}),$$

and let the totient of k, (supposed to contain i distinct prime factors) be expressed in the usual manner as the sum of 2^{i-1} positive terms P and the like number 2^{i-1} negative terms Q.

Then it may be proved (for it requires proof) that $\Sigma\phi P - \Sigma\phi Q$ will contain k; the quotient will contain the number of k-sided polygons in- and- exscribable about a cubic.

This theorem does not accord with the formula given by Professor Cayley in the Phil. Tr. for 1871, as quoted in the Math. Fortschr., Vol. III.

The number of triangles in- and- exscribable to a curve whose order is x, whose class is X and whose number of cusps + three times its class is ξ, is there stated to be

$$\begin{aligned} &X^4 + (2x^3-18x^2+52x-46)\,X^3 + (\overline{18}x^3+162x^2-420x+221)\,X^2 \\ &+ (52x^3-420x^2+704x+172)\,X + (x^4-46x^3+221x^2+172x) \\ &+ \xi\,\{\overline{9}X^2 + (\overline{12}x+135)\,X + (\overline{9}x^2+135x-600)\}. \end{aligned}$$

† When the cubic is $x^3+y^3+z^3$, X, Y, Z become $x^9+6x^6y^3+3x^3y^6-y^9, \ldots, xyz\,(x^6+x^3y^3+y^6)$ $X=0$ then gives $\frac{x^3}{y^3}=t-t^2$ if $t^3-3t+1=0$, *i. e.*, $t=2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$, $2\cos\frac{8\pi}{9}$; calling the three values of $\frac{x^3}{y^3}$ thus obtained $\tau_1\tau_2\tau_4$, one of the two real in- and- exscribed triangles will have at its vertices $\frac{x}{y}, \frac{y}{z}, \frac{z}{x} = \tau_1^{\frac13}, \tau_2^{\frac13}, \tau_4^{\frac13} = \tau_2^{\frac13}, \tau_4^{\frac13}, \tau_1^{\frac13} = \tau_4^{\frac13};\ \tau_1^{\frac12};\ \tau_2^{\frac13}$ respectively, and the triangle conjugate to it will have at its vertices $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$ equal to the same three systems of ratios.

† If $x^3+y^3+z^3+3mxyz$ be the given cubic, one set of 9 points will be found from the equation

$$[(1-\rho)\,y^3+(1-\rho^2)\,z^3]^3+27m^3\,(\rho y^6z^3+\rho^2y^3z^6)=0,$$

or

$$y^9-3\,((1-\rho^2)\,m^3-\rho^2)\,y^6z^3-((1-\rho)\,m^3-\rho)\,y^3z^6+z^9=0,$$

and the fellow set by interchanging y and z. The disadvantage of this method consists in its leading to equations with imaginary coefficients for finding *inter alia* real roots which the equations $Y=0$ or $Z=0$, being of odd degrees, show must necessarily always exist.

On making $x = 3$, $X = 6$ and $\xi = 18$ we ought to have 24 the number of in- and- exscribable triangles to a general cubic, but on making these substitutions the result will be found to be zero. It is *quite certain*, therefore, that this formula requires some correction which has been overlooked by its illustrious author. For I have actually, in the text, given a cubic and a triangle in- and- exscribable to it, not to add that it is manifestly impossible for a general cubic to refuse to pass under the form $xy^2 + yz^2 + zx^2 + mxyz$.

Before quitting this subject I wish to call attention to the fact that the formula above given for composite numbers is a form deduced from the form ϕk precisely as in the excursus, the expression for log χ_k^τ was deduced from log $(x^k - 1)$.* It is clear from general logical considerations that this sort of deduction must be continually liable to occur and a name is imperatively called for to express it as much as one was formerly wanted to express the kind of deduction which leads from an algebraical form to its Hessian. Here the deduction depends on the arithmetical constitution of the subject of the form, and it is a great impediment to the free course of ratiocination not to be able to pass at once, in language and in thought, from the form to its deduct. I intend then in future to call such deduct the *functional totient* of the form, say ϕk, from which it is derived, and to denote it by $(\phi\tau)\, k$. This constitutes a very important gain to arithmetical nomenclature.

I would further call attention to the fact of an arithmetical theorem of some considerable difficulty to demonstrate (by means of Fermat's extended theorem) in the general case as any one, who goes through the process of the proof for the single case of $k =$ the product of two primes, will easily satisfy himself, (I mean the theorem that the *functional totient* of $8\,(2^{k-1} - (\bar{1})^{k-1})\,(2^{k-2} - (\bar{1})^{k-2})$ is always divisible by k) should admit of an intuitional proof through the intervention of a pure property of cubic curves without any recourse to concepts drawn from reticulated arrangements, as in the applications of geometry to arithmetic made by Dirichlet and Eisenstein. This example of the possibility of such application (akin to that whereby the binomial theorem is made to prove that $\frac{\pi(m + m')}{\pi m \,.\, \pi m'}$ is an integer) is, as far as I can recall, without a precedent in mathematical history.

* The expression actually there given is for χ_k^τ and not its logarithm; using the notation explained above, and calling $\phi k = \log(x^k - 1)$ the cyclotomic of the 1st species to the index k, is $e^{(\phi\tau)\,k}$.

Postscriptum.

Mr. Franklin obtains my result as follows: The condition that the $(i-1)$th tangential shall lie on the first polar is of the degree $2.4^{i-1}+1$; the number of points on the cubic (exclusive of inflexions) satisfying this condition is $3(2.4^{i-1}+1)-27=24(4^{i-2}-1)$. But the $(i-1)$th tangential will be on the first polar, not only when it is a true antitangential, but also when it is the original point itself or the consecutive point; so that we have to deduct from the above number twice the number of points (exclusive of inflexions) whose $(i-1)$th tangentials are the points themselves; *i. e.* denoting by u_i the number of vertices of in- and- exscribed i-laterals, we have

$$\begin{aligned} a_i &= 24(4^{i-2}-1)-2u_{i-1} \\ &= 24\{2^{2i-4}-2^{2i-5}+\ldots+(-2)^{i-1}-(1-2+2^2-\ldots+(-2)^{i-3}\} \\ &= 8(2^{i-1}+(-1)^{i-2})(2^{i-2}-(-1)^{i-2}), \end{aligned}$$

which will be the number of the vertices, not only of true i-laterals, but also of all the $\frac{i}{\delta}$-laterals, (δ being any divisor of i except i itself) as well. Mr. Franklin further suggests that the discrepancy between this result for $i=3$ and Prof. Cayley's formula may be due to the latter not taking account of the peculiar kind of in- and- exscription in which the curve is in- and- exscribed at the same points. Finally, let us call the *summant* of a number k of the form $a^\lambda . b^\mu . c^\nu$ (a, b, c being primes) the well-known quantity consisting of $(1+\lambda)(1+\mu)(1+\nu)\ldots$ terms which represents the sum of the divisors of k. We may speak of a *functional summant* to ϕk obtained by prefixing ϕ to each monomial term in the *development* of the summant and denote it by $(\phi\sigma)k$. The equation $(\phi\sigma)k=\omega(k)$ has for its solution $fk=(\omega\tau)k$. My method gives at once, for the *functional summant* of u^k (*without exclusion* of inflexions) $(2^k-\tau^k)^2$, and accordingly, the functional totient to this form divided by k is the simplest expression for the number of ex- and- inscribed k-laterals to the cubic. Thus, for $k=1, 2, 3, 4, 5, 6$, that number is 9, 0, 24, 54, 216, 648 respectively.

2. *On 2 and 3 as cubic residues.*

For the benefit of those among my readers in this country who may not have access to the later works on arithmetic, it may be as well to point how with the aid of their Gauss or Legendre they may verify the conditions

which, later on, I shall have need to employ of 2 or 3 being cubic residues to k, a prime of the form $6i+1$. The cyclotomic function of the third degree in the variable to the index k, if we make $4k = m^2 + 27n^2$, is known to be $x^3 + x^2 - \frac{k-1}{2}x - \frac{3k-1+\varepsilon mk}{27}$, where $\varepsilon^2 = \pm 1$ and $m - \varepsilon$ contains 3. Connecting this with the same function formed in the manner in which the cyclotomics in the Excursus under Title 3 have been calculated, calling U the number of solutions of the congruence $1 + \beta + \gamma \equiv 0$ (mod. k), where β, γ are any two unequal cubic residues to k, and θ the number of solutions (1 or 0) of the congruence $1 + 2\beta \equiv 0$ (mod. k), it will easily be found, by comparing the constant terms in the two expressions, that

$$U + \frac{3\theta}{2} = \frac{k - 8 + \varepsilon m}{18}.$$

Hence, when $\theta = 1$, *i. e.* when 2 is a cubic residue, m (and therefore also n) must be even, and consequently when $\theta = 0$, or 2 is not a cubic residue, m must be odd, and *vice versâ*.

Again, if we compare the values of the sum of the 4th powers of the roots of the cyclotomic as found by the general method with that deducible from the given function, we shall find

$$V + \frac{2}{3}\vartheta = \frac{k^2 + 3k - 66 - 4m\varepsilon k}{162},$$

where V is the number of solutions of the congruence $1 + \beta + \gamma + \delta \equiv 0$, plus the number of solutions of the congruence $1 + \beta + 2\gamma \equiv 0$ (β, γ, δ being cubic residues to k) and ϑ the number of solutions of the congruence $1 + 3\beta \equiv 0$ (mod. k), *i. e.* 1 or 0, according as 3 is, or is not, a cubic residue to k.

The numerator is necessarily divisible by 54, but the criterion of ϑ being 0 or 1 depends on its being divisible or not by 81. On substituting for k its value in terms of m and n, it will be found that 16 times the numerator to modulus 81 is congruous with 54 times $(n^2 - 1) + \varepsilon\left\{\left(\frac{m-\varepsilon}{3}\right)^3 - \frac{m-\varepsilon}{3}\right\}$, and consequently is divisible or not by 81 according as n is not, or is, divisible by 3. Hence $\vartheta = 1$ when n is divisible by 3 and otherwise is 0.

The joint effect of these two results may be translated into the following statement, which is better adapted than the more complete* form of enunciation would be to the purposes of this memoir.

*I mean more complete in the sense of fixing the cubic character in the case of 3 being a non-residue, which is unimportant to the matter in hand.

If $k = f^2 + 3g^2$, *when* $(f \pm g)$ *contains* 9, 3 *is, and* 2 *is not, a cubic residue; when* g *contains* 3, *but not* 9, 2 *is, and* 3 *is not, a cubic residue*; *when* g *contains* 9, 2 *and* 3 *are each of them cubic residues, and in any other case neither* 2 *nor* 3 *is a cubic residue to* k.*

The equation $U + \frac{3\theta}{2} = \frac{3k - 1 + \varepsilon mk}{18}$ contains a complete solution of the interesting question, "How many times, if the cubic residues to a given modulus are set out in a regular ascending series, will consecutive terms differ from one another by a single unit?" When 2 is not a cubic residue, the answer is obviously $2U$, for $1 + \alpha + \beta = n$ gives two sequences, α, $n - \beta$ and β, $n - \alpha$, differing by units. But when 2 is a cubic residue, there will be three extra sequences not contained among the $2U$ just spoken of, viz:

$$1, 2; \quad \frac{k-1}{2}, \frac{k+1}{2}; \quad k-2, k-1.$$

Hence, in each case, the number is $2U + 3\theta$, *i. e.* $\frac{k - 8 + \varepsilon m}{9}$, or, if we count in 0 as a residue, $\frac{k + \varepsilon m + 1}{9}$.

SECTION 2.

On certain numbers and classes of numbers that cannot be resolved into the sum or difference of two rational cubes.

Title 1. *Theorem on irresoluble numbers whose prime factors other than* 2 *or* 3 *are of the form* $18n + 5$ *or* $18n + 11$.† I propose to prove the following collective theorem. If A represents any one of the numbers 1, 2, 3, 4, 18, 36 or any number of the form

$$\begin{array}{llll} p, & q, & p^2, & q^2, \\ 9p, & 9q, & 9p^2, & 9q^2, \\ 2p, & 4q, & 4p^2, & 2q^2, \\ pq, & p_1p_2^2, & q_1q_2^2, & p^2q^2, \end{array}$$

(where any p means a prime number of the form $18n + 5$, and any q a prime of the form $18n + 11$) A will be irresoluble into the sum of two unequal rational cubes.

*In other words, if $4p = m^2 + 27n^2$ [an equation always possible when $p = 6i + 1$], n divisible by 2 is the necessary and sufficient condition of 2, and n divisible by 3 is the necessary sufficient condition of 3, being a cubic residue to p.

† This theorem includes and transcends all the cases of irresolubility that had been discovered prior to the date of publication of the Proem in the last number of the Journal, with the exception of certain specific numbers whose irresolubility had been determined by the Abbé Pépin.

Lemma. If we decompose A (when it is not a prime) into any factors f, g, h, prime to each other, other than 1, 1, A, the equation $fx^3 + gy^3 + hz^3 = 0$ will be irresoluble in integers.

I prove this by showing that the above equation converted into a congruence to *modulus* 9 is irresoluble in integers.

x^3, y^3, z^3, each of them to this modulus is equivalent to one or the other of the three numbers $\bar{1}$, 0, 1.

p, p_1, p_2	to	this modulus	is equivalent	to	$\bar{4}$
q, q_1, q_2	"	"	"	"	2
p^2, p^2, p_2^2	"	"	"	"	$\bar{2}$
q, q_1^2, q_2^2	"	"	"	"	4,

and on inspection, it will easily be verified that the limited linear congruence $f\lambda + g\mu + h\nu \equiv 0$ [mod. 9], where λ, μ, ν must each be picked out of the three numbers $\bar{1}$, 0, 1, has no solution.

Hence, if $fx^3 + gy^3 + hz^3 = 0$ and $f . g . h = A$, and x, y, z are supposed to be prime to each other, two of the quantities f, g, h will be unities and the third equal to A.

Let, now, $x^3 + y^3 + Az^3 = 0$ be supposed soluble in integers. Then, since A contains no $6n + 1$ prime, we must have

$$\left.\begin{array}{l} x + y = A\zeta^3 \\ x^2 - xy + y^2 = \omega^3 \\ z = -\zeta\omega \end{array}\right\} \text{when } x + y \text{ does not contain } 3,$$

and

$$\left.\begin{array}{l} x + y = 9A\zeta^3 \\ x^2 - xy + y^2 = 3\omega^3 \\ z = -3\zeta\omega \end{array}\right\} \text{when } x + y \text{ contains } 3.$$

If $x + y$ is even, since $x^2 - xy + y^2 = \left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2$, we must have $\frac{x+y}{2} + \sqrt{-3}\,\frac{x-y}{2} = (\xi + \sqrt{-3}\eta)^3$, when $x + y$ does not contain 3, and $\frac{x-y}{2} + \sqrt{-3}\,\frac{x+y}{6} = (\xi + \sqrt{-3}\eta)^3$, when $x + y$ contains 3. In the one case $\frac{x+y}{2} = \xi^3 - 9\eta^3$, $\frac{x-y}{2} = 3\xi^2\eta - 3\eta^3$, and in the other $\frac{x-y}{2} = \xi^3 - 9\eta^2\xi$, $\frac{x+y}{6} = 3\xi^2\eta - 3\eta^3$.

In the one case, then, $2\xi(\xi - 3\eta)(\xi + 3\eta) = A\zeta^3$, and in the other $2\eta(\xi - \eta)(\xi + \eta) = A\zeta^3$. In either case, therefore, there is an equation-system

of the form $\rho\sigma\tau = -A\zeta^3$, $\rho + \sigma + \tau = 0$, to be satisfied; therefore, disregarding permutations of ρ, σ, τ, we must have

$$\rho = fx_1^3, \quad \sigma = gy_1^3, \quad \tau = hz_1^3$$
$$f \cdot g \cdot h = A, \; x_1y_1z_1 = -\zeta$$
$$fx_1^3 + gy_1^3 + hz_1^3 = 0,$$

and consequently by the Lemma $x_1^3 + y_1^3 + Az_1^3 = 0$ (or the same equation with x_1, y_1, z_1 interchanged) where $x_1y_1z_1$ is a factor of z.

Continuing the same process perpetually, as long as the new x and y have the same parity, each new x, y, z being contained in the immediately preceding z, must perpetually decrease, and if the process could be indefinitely continued, x and y must each evidently become unity, since otherwise z could go on decreasing without limit. This could only happen when $A = 2$, and even then is excluded by the condition that the cubes are to be unequal as well as rational.* Hence, if the proposed equation is soluble at all, it must contain solutions in which x and y are one even and the other odd.

On this hypothesis, let us consider separately case (1), where $x + y$ does not, and case (2) where $x + y$ does contain 3.

Case (1). Here $(x + y)^2 + 3(x - y)^2 = 4(L^2 + 3M^2) = 4\omega^3$, and all the solutions of this equation are necessarily included in those of the system $L^2 + 3M^2 = \omega^3$, $x + y = L + 3M$, $x - y = L - M$.

Hence $x + y = \xi_1^3 + 9\xi_1^2\eta_1 - 9\eta_1^2\xi_1 - 9\eta_1^3 = A\zeta^3$. On making $\xi_1 = \xi - 3\eta_1$, this becomes $\xi^3 - 36\xi\eta_1^2 + 72\eta_1^3 = A\zeta^3$, or, making $\eta' = 6\eta_1$, $3\xi^3 - 3\xi\eta'^2 + \eta'^3 = 3A\zeta^3$, which, on writing $\eta' = \eta + \xi$, becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3A\zeta^3$, where A unless it is unity contains at least one factor that is not of the form $18n \pm 1$, or else (in the case when $A = 3$) the square of 3. Hence, by virtue of the cyclotomic law for index 9, species 2 (conjugate class) (see Table, p. 367), the above equation is insoluble in integers.†

*To prove this. Let ξ, η, ζ be the system of variables, for which $\xi = 1$, $\eta = 1$ and x, y, z the system immediately preceding it. Then we have $A = 2$, $\xi = 1$, $\eta = 1$, $\zeta = -1$, and either $x - y = 0$, or $x + y = 0$. The latter of these equations would imply $z = 0$ and the former $x : y : z :: 1 : 1 : -1$, and so continually until we fall back on the original equation in x, y, z. Hence the only possible resolution of 2, if $x + y$ is even, is into two equal cubes.

† $3A$ not containing any cube, ξ and $3A$ must be prime to each other, since otherwise η, ξ, ζ would have a common measure. Hence we may make $\eta = \xi\mu - 3A\lambda$, and, consequently, $(\mu^3 - 3\mu + 1)\xi^3 \equiv 0$ mod. $3A$, and, therefore, $\mu^3 - 3\mu + 1$ must contain $3A$.

This conclusion would not hold if $3A$ were of the form A_1B^3 where A_1 contained no cube. We could then only infer $\mu^3 - 3\mu + 1 \equiv 0$ mod. A_1. Thus, in the case of $A = 9$, $3A = B^3$, and our inference would become $\mu^3 - 3\mu + 1 \equiv 0$ mod. 1, which, of course, is satisfied, and, accordingly 9 ought to be resoluble into two cubes, as it obviously is, viz: into 1 and 8. Thus, the equation $x^3 - 3xy^2 + y^3 = 3Az^3$, when $A = 9$ has an infinite number of solutions when $A = 3$ has no solution, and when $A = 1$ has just 3 solutions.

Case (2). Here, using L and M in the same sense as above, $\frac{x+y}{3} = L - M$ and $x-y=L+3M$ or $\xi_1^3 - 3\xi_1^2\eta_1 - 9\xi_1\eta_1^2 + 3\eta_1^3 = 3A\zeta^3$. Here writing $2\eta_1 = -\xi$, $\xi_1 = \eta + 2\xi$, the equation becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3A\zeta^3$, and is insoluble in integers as before. Hence, since by hypothesis $x+y$ is not even, and it has been shown that it cannot be odd, *the number A when not unity is irresoluble into the sum or difference of two unequal rational cubes.**

When A is unity the equation above written becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3\zeta^3$, the necessity for discussing what may be avoided by choosing the x, y out of x, y, z (which in this case are indistinguishable) so as to make $x+y$ always even, which is the ordinary and easier method; but it is not without interest to show how the desired conclusion may be arrived at by keeping $x+y$ always odd. This may be done as follows: The equation between ξ, η, ζ, on writing $\eta + \zeta = u$, $\zeta - \xi = v$, $-\eta + \xi + \zeta = w$† becomes $uv^2 + vw^2 + wu^2 = 0$ which, as shown in foot note to p. 383, involves the relations $u = y'^2 z'$, $v = z'^2 x'$, $w = x'^2 y'$ and consequently $x'^3 + y'^3 + z'^3 = 0$ where $x'y'z' = \sqrt[3]{uvw}$.

Let us use in general two or more separate letters enclosed within a parenthesis to denote the absolute value of the *greatest one of them* (their *dominant* as I am wont to call it).

When $x+y$ does not contain 3, $x + y = \zeta^3$, $x^2 - xy + y^2 = (\xi_1^2 + 3\eta_1^2)^3$. Hence $\zeta < 2^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$ $(\xi_1, \eta_1) < 3^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$. Therefore $(\xi_1, \eta_1, \zeta) < 3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$, and consequently since $\xi = \xi_1 + 3\eta_1$ and $\eta = -\xi_1 + 3\eta_1$, $(\xi, \eta, \zeta) < 4 \,.\, 3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$ and therefore $(u, v, w) < 4 \,.\, 3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}$. Hence $x' . y' . z' < (u, v, w) < 4 \,.\, 3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}$.

In like manner when $x+y$ does contain 3, from the equations $\xi = -2\eta_1$, $\eta = \xi_1 - \eta_1$, $x + y = 9\zeta^3$, $x^2 - xy + y^2 = 3(\xi_1^2 + 3\eta_1^2)^3$, follow $\zeta < \left(\frac{1}{3}\right)^{\frac{2}{3}}(x, y)^{\frac{1}{3}}$ $(\xi_1, \eta_1) < (x, y)^{\frac{1}{3}}$, $(\xi_1, \eta_1, \zeta) < (x, y, z)^{\frac{1}{3}}$, $(\xi, \eta, \zeta) < (x, y, z)^{\frac{1}{3}}$, $x' . y' . z' < (u, v, w) < 3(x, y, z)^{\frac{1}{3}}$.

In any case therefore $x' . y' . z' < 4 \,.\, 3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}} < 18(x, y, z)^{\frac{1}{3}}$. But the difference between any two cubes except 8 and 1 being greater than 8, the

It may be worth noting that, in general, if $(x, y)^n = Az^n$, and $A = A_1 B^n$, where A_1 contains no nth power of a number $(x, 1)^n$ will contain A_1 as a divisor, provided that the coefficient of x^n in $(x, y)^n$ is prime to A_1. Cases of this inference being drawn of course frequently occur, but the general principle, obvious as it is, I do not recollect to have seen formulated in the text books. It may be made more precise by the statement that any factor of A_1 prime to the coefficient of x^n will be a divisor of $(x, 1)^n$.

* The equations of substitution are: for case 1, $\xi = \xi_1 + 3\eta_1$, $\eta = -\xi_1 + 3\eta_1$; and for case 2, $\xi = -2\eta_1$, $\eta = \xi_1 - \eta_1$.

† From these equations it is obvious that the dominant, *i. e.* the arithmetically greatest of the quantities u, v, w, is less than 3 times the dominant of ξ, η, ζ.

smallest of the numbers x', y', z' cannot be less than 3, and, since neither 3^3+4^3 nor 3^3+5^3 is a cube, it follows that $\frac{x'.y'.z'}{(x',y',z')}>18$, and therefore $(x', y', z') < (x, y, z)^{\frac{1}{3}}$, or the dominant of the quantities x, y, z which satisfy $x^3+y^3+z^3=0$ is continually replaced by another similar dominant less than the cube root of its predecessor, which is impossible.

Hence $x^3+y^3+z^3=0$ is insoluble. Let us see how this is reconcilable with the existence of the 3 rational solutions of $\eta^3-3\eta\xi^2+\xi^3=3A\zeta^2$, viz: $\xi, \eta, \zeta=\bar{1}, 1, 1$ or $2, 1, 1$ or $1, 2, \bar{1}$ respectively.

In case (1) $\xi=\xi_1+3\eta_1 \quad \eta=-\xi_1+3\eta_1 \quad \xi, \eta=\bar{1}, 1$ gives $\eta_1=0$ $\xi, \eta=2, 1$ gives $\eta_1=-\xi_1$ $\xi, \eta=1, 2$ gives $\eta_1=\xi_1$. In each instance therefore $M=3\eta_1(\xi_1^2-\eta_1^2)=0$ and consequently $x+y=L=x-y$ and $y=0$.

In case (2) $\xi=-2\eta_1 \quad \eta=\xi_1-\eta_1 \quad \xi, \eta=\bar{1}, 1$ gives $\xi_1=3\eta_1$ $\xi, \eta=2, 1$ gives $\xi_1=-3\eta_1$ and $\xi, \eta=1, 2$ gives $\xi_1=0$.

In each instance therefore $L=\xi_1(\xi_1^2-9\eta_1^2)=0$ and therefore $x=0$. Thus the rational solutions of the equation in ξ, η, ζ in both cases correspond to rational but futile solutions of the equation in x, y, z.

[To be continued.]

A Quincuncial Projection of the Sphere.

By C. S. Peirce.

[Published by the authority of the Superintendent of the U. S. Coast and Geodetic Survey.]

For meteorological, magnetological and other purposes, it is convenient to have a projection of the sphere which shall show the connection of all parts of the surface. This is done by the one shown in the plate. It is an orthomorphic or conform projection formed by transforming the stereographic projection, with a pole at infinity, by means of an elliptic function. For that purpose, l being the latitude, and θ the longitude, we put

$$\cos^2 \phi = \frac{\sqrt{1 - \cos^2 l \cos^2 \theta} - \sin l}{1 + \sqrt{1 - \cos^2 l \cos^2 \theta}},$$

and then $\frac{1}{2} F\phi$ is the value of one of the rectangular coördinates of the point on the new projection. This is the same as taking

$$\cos am\,(x + y\sqrt{-1})(\text{angle of mod.} = 45^\circ) = \tan \frac{p}{2} (\cos \theta + \sin \theta \sqrt{-1}),$$

where x and y are the coördinates on the new projection, p is the north polar distance. A table of these coördinates is subjoined.

Upon an orthomorphic potential the parallels represent equipotential or level lines for the logarithmic projection, while the meridians are the lines of force. Consequently we may draw these lines by the method used by Maxwell in his Electricity and Magnetism for drawing the corresponding lines for the Newtonian potential. That is to say, let two such projections be drawn upon the same sheet, so that upon both are shown the same meridians at equal angular distances, and the same parallels at such distances that the ratio of successive values of $\tan \frac{p}{2}$ is constant. Then, number the meridians and also the parallels. Then draw curves through the intersections of meridians with meridians, the sums of numbers of the intersecting meridians being constant on any one curve. Also, do the same thing for the parallels. Then these curves will represent the meridians and parallels of a new projection having north poles and south poles wherever the component projections had such poles.

Functions may, of course, be classified according to the pattern of the projection produced by such a transformation of the stereographic projection with a pole at the tangent points. Thus we shall have—

1. Functions with a finite number of zeros and infinites (algebraic functions).
2. Striped functions (trigonometric functions). In these the stripes may be equal, or may vary progressively, or periodically. The stripes may be simple, or themselves compounded of stripes. Thus, sin (a sin z) will be composed of stripes each consisting of a bundle of parallel stripes (infinite in number) folded over onto itself.
3. Chequered functions (elliptic functions).
4. Functions whose patterns are central or spiral.

I. *Table of Rectangular Coördinates for Construction of the "Quincuncial Projection."*

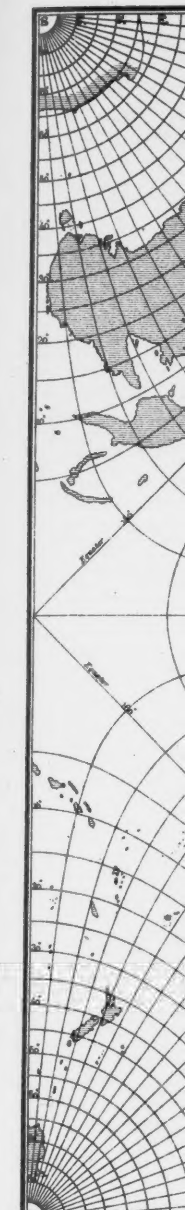
x (for longitudes in upper line). y (for longitudes in lower line.)

LAT.	0° 90	5° 85	10° 80	15° 75	20° 70	25° 65	30° 60	35° 55	40° 50	45° 45	50° 40	55° 35	60° 30	65° 25	70° 20	75° 15	80° 10	85° 5	LAT.
85°	.033	.033	.033	.032	.031	.030	.029	.027	.025	.024	.021	.019	.017	.014	.011	.009	.006	.003	85°
80	.067	.066	.066	.064	.063	.061	.058	.055	.051	.047	.043	.038	.033	.028	.023	.017	.012	.006	80
75	.100	.100	.099	.097	.094	.091	.087	.082	.077	.071	.065	.058	.050	.042	.034	.026	.017	.009	75
70	.135	.134	.133	.130	.127	.122	.117	.110	.103	.095	.087	.077	.067	.057	.046	.035	.023	.012	70
65	.169	.169	.167	.163	.159	.154	.147	.139	.130	.120	.109	.097	.085	.072	.058	.044	.029	.015	65
60	.205	.204	.201	.198	.192	.185	.177	.168	.157	.145	.131	.117	.102	.086	.070	.053	.036	.018	60
55	.241	.240	.237	.232	.226	.218	.208	.197	.184	.170	.154	.138	.120	.102	.082	.062	.042	.021	55
50	.278	.277	.274	.269	.261	.251	.240	.227	.212	.196	.178	.159	.139	.117	.095	.072	.048	.024	50
45	.317	.316	.312	.306	.297	.286	.273	.258	.241	.223	.202	.181	.158	.134	.109	.083	.055	.028	45
40	.357	.356	.351	.344	.334	.321	.307	.290	.270	.250	.228	.204	.179	.151	.123	.094	.063	.032	40
35	.400	.398	.393	.384	.373	.358	.341	.322	.301	.279	.254	.228	.200	.170	.139	.106	.071	.036	35
30	.446	.443	.437	.427	.413	.396	.377	.356	.332	.308	.281	.253	.222	.190	.155	.119	.081	.041	30
25	.495	.492	.484	.471	.455	.435	.414	.391	.365	.338	.309	.279	.246	.211	.174	.134	.091	.046	25
20	.548	.545	.534	.518	.498	.476	.452	.426	.398	.369	.339	.307	.272	.235	.195	.151	.104	.053	20
15	.609	.604	.589	.568	.544	.517	.490	.461	.432	.401	.369	.336	.300	.262	.219	.173	.121	.062	15
10	.681	.672	.649	.620	.590	.559	.528	.497	.466	.434	.401	.367	.330	.291	.248	.200	.143	.076	10
5	.775	.752	.713	.673	.635	.600	.566	.532	.500	.467	.433	.399	.363	.324	.282	.234	.177	.102	5
0	1.000	.841	.774	.723	.679	.639	.602	.567	.533	.500	.467	.433	.398	.361	.321	.277	.226	.159	0

II. *Preceding Table Enlarged for the Spaces Surrounding Infinite Points.*

x (for longitudes in upper line). y (for longitudes in lower line).

LAT.	0°	1°	2°	3°	4°	5°	6°	8°	10°	12½°	15°	75°	77½°	80°	82°	84°	85°	86°	87°	88°	89°	LAT.
	90	89	88	87	86	85	84	82	80	77½	75	15	12½	10	8	6	5	4	3	2	1	
15°	.609	.609	.608	.607	.606	.604	.602	.596	.589	.579	.568	.173	.147	.121	.098	.074	.062	.050	.038	.025	.013	15°
12½	.643	.643	.642	.641	.639	.636	.634	.627	.618	.606	.594	.185	.159	.131	.107	.082	.069	.055	.042	.028	.014	12½
10	.681	.681	.680	.678	.675	.672	.668	.659	.649	.635	.620	.200	.173	.143	.118	.091	.076	.062	.047	.031	.016	10
8	.715	.714	.713	.710	.706	.702	.697	.686	.674	.658	.641	.213	.185	.155	.129	.100	.085	.069	.052	.035	.018	8
6	.753	.752	.750	.746	.741	.735	.728	.714	.700	.681	.662	.227	.199	.169	.142	.112	.095	.078	.060	.040	.020	6
5	.775	.774	.770	.765	.759	.752	.745	.729	.713	.692	.673	.234	.207	.177	.150	.119	.102	.084	.065	.044	.022	5
4	.798	.797	.793	.786	.779	.770	.761	.743	.725	.704	.683	.242	.215	.185	.158	.128	.110	.092	.071	.049	.025	4
3	.825	.823	.817	.808	.798	.788	.778	.757	.738	.715	.693	.250	.224	.194	.168	.137	.120	.101	.079	.055	.029	3
2	.857	.853	.843	.831	.819	.806	.794	.772	.750	.726	.703	.259	.233	.204	.178	.148	.131	.112	.090	.065	.035	2
1	.899	.889	.872	.854	.839	.824	.810	.785	.763	.737	.713	.268	.243	.215	.190	.161	.144	.126	.105	.079	.046	1
0	1.000	.929	.899	.877	.857	.841	.825	.798	.774	.747	.723	.277	.253	.226	.202	.175	.159	.143	.123	.101	.071	0



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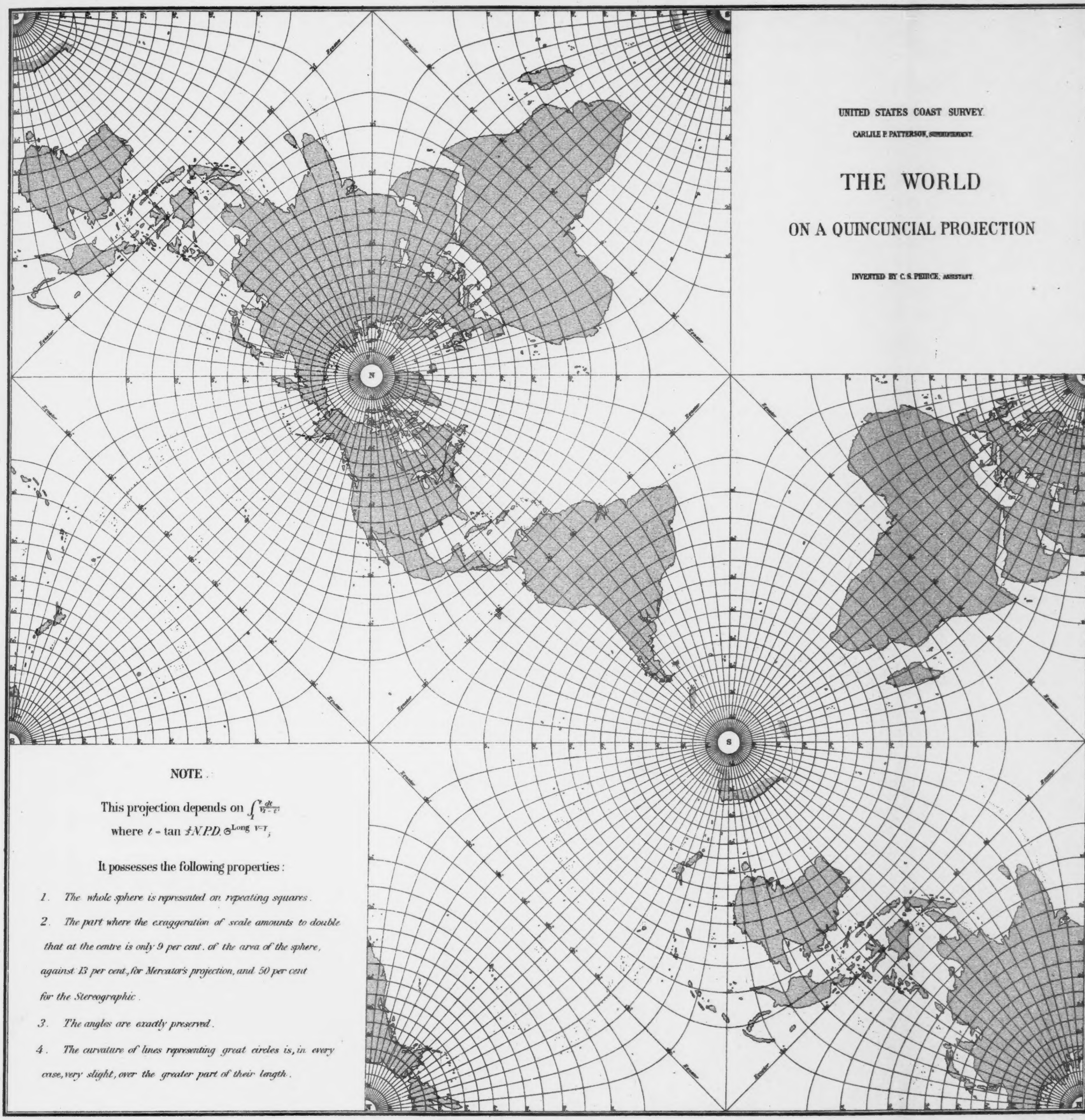
UNITED STATES COAST SURVEY.
CARLILE P. PATTERSON, SUPERINTENDENT.
THE WORLD
ON A QUINCUNCIAL PROJECTION
INVENTED BY C. S. PEIRCE, ASSISTANT.
NOTE.
This projection depends on $\int_{1}^{x} \frac{dt}{\sqrt{1-t^4}}$
where $t = \tan \frac{1}{2} N.P.D. \, \Theta^{\text{Long} \sqrt{-1}}$;
It possesses the following properties:
1. The whole sphere is represented on repeating squares.
2. The part where the exaggeration of scale amounts to double that at the centre is only 9 per cent. of the area of the sphere, against 13 per cent. for Mercator's projection, and 50 per cent for the Stereographic.
3. The angles are exactly preserved.
4. The curvature of lines representing great circles is, in every case, very slight, over the greater part of their length.

Notes on the "15" Puzzle.

I.

By Wm. Woolsey Johnson, *Annapolis, Md.*

The puzzle described below has recently been exercising the ingenuity of many persons in Baltimore, Philadelphia and elsewhere. A ruled square of 16 compartments is numbered as in this diagram:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

the 16th square being left blank. Fifteen counters, numbered in like manner, are placed at random upon the squares so that one square is vacant. The counter occupying any adjacent square may now be moved into the vacant square—thus: If No. 7 is vacant, either of the counters occupying Nos. 3, 6, 8, 11 can be moved into it, but no diagonal move is allowed. The puzzle is to bring all the counters into their proper squares by successive moves.

It seems to be generally supposed, by those who have tried the puzzle, that this is always possible, whatever be the original random position of the counters, but this is an error, as the following demonstration will show:

When the blank or sixteenth square is the vacant one, the arrangement of the counters may be called a positive or negative one, according as the term of the 15-square determinant, which has for first and second subscripts the numbers on the squares and counters, is positive or negative. Let n

moves be made, leaving some other square vacant, and then suppose the counter which occupies the blank square to be transferred directly to the vacant square, we thus obtain a positive or a negative arrangement. Had $n+1$ moves been made before the transfer took place, the arrangement produced would have been one which can be derived from that last mentioned by a single interchange of two counters. (For example, if the $(n+1)$th move is from No. 6 to No. 7, the moved counter will be in No. 7 and the transferred counter in No. 6; whereas, had the transfer taken place after n moves, the former would have been in No. 6 and the latter in No. 7).

Now the first two moves followed by a transfer are equivalent to one interchange; therefore the displacement effected by n moves followed by a transfer is one which could have been produced by $n-1$ interchanges. Now suppose that after m moves the blank space is again left vacant, then the mth move is itself a transfer from the blank square, and therefore the displacement produced by the m moves is one which might have been produced by $m-2$ interchanges.

If the squares were coloured, as in a chess board, each move would change the colour of the vacant square, and therefore m is an even number; it follows that the displacement is one which might have been produced by an even number of interchanges, and can never change a positive to a negative arrangement or the reverse; hence the desired arrangement, which is a positive one, can never be produced if the original random arrangement happens to be a negative one. This conclusion is obviously not affected in any way by the shape of the board.

In order to make this demonstration satisfactory to non-mathematicians who may be interested in this puzzle, I add a simple demonstration of the theorem upon which the classification of the arrangements as positive and negative depends, viz: that a permutation that can be derived from a given one by an odd number of interchanges can never be produced by an even number of interchanges. Let the numbers $1, 2, \ldots n$ be written down in natural order, and under them place any other permutation of the same numbers, thus if $n = 15$, as in the present case, we might have

$$\begin{array}{ccccccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15; \\ 12, & 10, & 8, & 15, & 9, & 6, & 3, & 4, & 11, & 1, & 7, & 2, & 14, & 5, & 13. \end{array}$$

Now group the numbers into cycles as follows: beginning with any one of the numbers, look for it in the upper row and write next to the number that

which is found under it, then looking for the latter in the upper row, write next the number found under it, and so on until we find the original number in the lower row. Thus in the example above beginning with 1, we have the cycle

1, 12, 2, 10.

Then taking a number not found in this cycle, say 3, form a new cycle, and so on till the numbers are exhausted. In this case we shall find the other cycles to be

3, 8, 4, 15, 13, 14, 5, 9, 11, 7, and 6,

the last cycle happening to consist of a single number. Let m denote the number of these cycles. In the above case $m = 3$. Now let two of the lower numbers be interchanged. A little consideration will show that if these numbers belong to the same cycle, this cycle will be broken up into two cycles; but if they belong to different cycles, these cycles will be combined into a single one. In either case, the value of m will be changed from an even to an odd number, or the reverse. The same is true of the number $n - m$. Now, when the lower numbers are in natural order, there are n cycles, each composed of a single number, and $n - m = 0$. Hence, starting from this arrangement, any odd number of interchanges will produce an arrangement in which $n - m$ is odd, and any even number, one in which $n - m$ is even. The former are the negative, and the latter the positive arrangements alluded to above.

Postscript.

Since the above was written the puzzle has been published in the form of a square box containing 15 blocks, the squares *not being numbered.* The requirement is simply to "move the blocks until in regular order." It has been shown in the New York Evening Post that when it is impossible to arrange the blocks, with the block 1 in a certain corner, it is possible to obtain a regular arrangement with the block 1 in an adjacent corner.

II.

BY WILLIAM E. STORY.

IN the preceding note Mr. Johnson has proved that, with the ordinary form of the puzzle, a positive arrangement with the 16th square blank cannot

be obtained from a negative arrangement with the same square blank. But, evidently, the same method of proof will shew that, on a square or rectangular board divided by any number of vertical and any number of horizontal lines into spaces or squares, if a number of counters one less than that of squares, numbered successively from 1 on, be arranged in any way, and then moved as in the "15" puzzle, a positive arrangement cannot be converted by such moves into a negative arrangement with the same square blank, nor *vice versâ*. And this result is entirely independent of the position of the blank square. Moreover we may, in forming the arrangements of the numbers of the counters, take the first number from any given square of the board, the second from any other, the third from any remaining square, and so on, without affecting the validity of the proof, provided we use the squares in the same order in all the arrangements considered. The order in which I shall suppose the squares to be employed in forming the arrangements is this: beginning at the left-hand square of the upper row, I shall take the squares in succession along the upper row from left to right, then back along the second row from right to left, and so on along the successive rows, alternately from left to right and from right to left, until all the squares on the board have been taken, omitting the vacant square. The succession of the numbers of the counters taken in this order we shall speak of simply as the *order of the arrangement*, calling the order positive or negative according as the numbers taken in this way form a positive or negative permutation. That arrangement whose order is the natural order of the numbers we will call the *standard arrangement*. We proceed now to deduce a rule for determining whether a given arrangement can be converted into the standard arrangement, and, if so, in what manner this can be effected.

1. Evidently, with any given arrangement, two squares, upon which are counters adjacent in the order of the arrangement, are either adjacent squares upon the board, or both adjacent to the blank space. Now the blank can be interchanged with any adjacent counter by simply moving the latter into the place of the former. Thus the blank can be made to pass from any one position on the board to any other, by successive interchanges with an adjacent counter, without altering the order of the arrangement. Thus the condition for the possibility of converting any given arrangement into the standard arrangement may be treated as independent of the position of the blank square, but depending only upon the order of the given arrangement, since

the arrangement itself depends only upon the position of the blank square and the order.

2. In any arrangement any counter can be made to pass over the two counters next preceding or next succeeding it in order, without otherwise altering the order; *i. e.* if a, b, c be any three successive counters in the order of the arrangement, then a may be passed over b and c. For, bring the blank space to the side of c, so that a, b, c and the blank shall occupy four successive squares, situated in two successive rows (or in one, in which case either adjacent row may be taken for the second), and these two rows, joined at both ends, form a closed circuit in which a, b, c may be moved along until they, together with the blank space, occupy the two squares nearest the same end of both rows, thus:

a	b
	c

or

	c
a	b

.

If then a, b, c, a be successively moved into the blank space, the arrangement in these four squares becomes

b	c
	a

or

	a
b	c

,

in the one or the other case respectively, which is the above arrangement with a, b, c respectively replaced by b, c, a. Reversing the moves by which the three counters were brought to the end of the rows, and also those by which the blank square was brought to the side of c, we have the original arrangement with a, b, c respectively replaced by b, c, a, *i. e.* a has been passed over b and c, but no other change made in the order. It is evidently not necessary to reverse the moves by which the blank was brought to the side of c, for these do not affect the *order* of the arrangement.

Whatever be the given arrangement, the counter marked 1 may be passed over the counters preceding it in order, two at a time, until it occupies

either its proper position in the standard arrangement (viz: the first square on the board), or the next square; if, in this process, it comes into the square adjacent to its own, the counter which occupies its square may be passed over it and the next counter, thus leaving it in its proper place. When the 1 is in place, we may pass up the 2 and each successive number, passing it back over two counters at a time until it reaches its own square or the next; if the latter, the counter in question may be brought into its own square by causing the counter which occupies its place to pass over it and the next. This process may be continued until only the last two counters remain, when these will be either in their proper or in inverted order. Thus every arrangement may be brought into one or the other of these two *final* arrangements, differing by one interchange, and therefore of opposite characters (the first of a positive and the second of a negative order). From which it follows (since no arrangement whose order is positive can be changed into one whose order is negative, or *vice versâ*) that *every arrangement whose order is positive, and only such, can be converted into the standard arrangement.* This is the desired criterion for the possibility of a *standard solution.*

It is evident that any two arrangements, which can be converted into the same third arrangement, can be converted into each other, and that any two arrangements cannot be converted into each other, if they can be converted into two other arrangements not convertible into each other. Now every arrangement can be converted into one or the other of the two above-mentioned final arrangements. Hence any two arrangements are interchangeable if their orders are both positive or both negative, and not interchangeable otherwise. Hence, also, an arrangement whose order is positive can or cannot be converted into a given arrangement, according as the latter is convertible into the standard arrangement by an even or an odd number of interchanges; and an arrangement whose order is negative can or cannot be converted into a given arrangement, according as the latter is convertible into the standard arrangement by an odd or an even number of interchanges. Now what may be called the *natural arrangement* (*i. e.* the arrangement in which the numbers on the counters follow each other from left to right in the upper, second, third, etc. row in their natural order, and the right hand square of the bottom row is blank) can be converted into the standard arrangement by reversing the order of the counters in the second, fourth and every even row. Evidently, any row may be reversed by inter-

changing the counters equally distant from its two ends. Thus a row containing an even number of counters may be reversed by a number of interchanges equal to half the number of counters in the row, and a row containing an odd number of counters by half the number less one, since the position of the middle counter is not altered by reversing. Hence the number of interchanges necessary to reverse a row of an odd number of counters is the same as for a row containing a number of counters one less. The number of necessary interchanges is the same for each even row, unless the board contains an even number of rows and an even number of columns, in which case the number of interchanges for the last row will be one less than for any other even row.

Representing the number of rows on the board by r and the number of columns by c, we shall have four cases, viz:

I. r even, c even; II. r even, c odd; III. r odd, c even; IV. r odd, c odd.

The number of interchanges necessary to convert the natural arrangement into the standard arrangement in each case is

I. $\frac{1}{2}c$ in each of $\frac{1}{2}r - 1$ rows and $\frac{1}{2}c - 1$ in one row,

II. $\frac{1}{2}(c-1)$ in each of $\frac{1}{2}r$ rows,

III. $\frac{1}{2}c$ in each of $\frac{1}{2}(r-1)$ rows,

IV. $\frac{1}{2}(c-1)$ in each of $\frac{1}{2}(r-1)$ rows;

i. e. I. $\frac{1}{4}cr - 1$, II. $\frac{1}{4}(c-1)r$, III. $\frac{1}{4}c(r-1)$, IV. $\frac{1}{4}(c-1)(r-1)$.

We may divide all possible boards into two classes, regarding as of the *first class* a board for which the number just found is even, and as of the *second class* one for which this number is odd. We have then this rule:

On a board of the first class a given arrangement can or cannot be converted into the natural arrangement, according as its order is even or odd; but on a board of the second class a given arrangement can or cannot be converted into the natural arrangement, according as its order is odd or even.

For the ordinary "15" puzzle we have $r=4$, $c=4$, which belong to Case I.; $\frac{1}{4}cr - 1 = 3$, which being an odd number, the board is of the second kind, and the natural arrangement can be obtained from any arrangement whose order is odd, but not from one whose order is even. For a square board with five rows and five columns we have $r=5$, $c=5$, belonging to Case IV., $\frac{1}{4}(c-1)(r-1) = 4$, and the board is of the first class, hence the natural arrangement can be obtained from any arrangement whose order is even, but not from one whose order is odd. We may designate as the *reversed*

natural arrangement that which is obtained from the natural arrangement by reversing *all* the rows, leaving the left-hand square of the lower row blank. Using the notation just employed, the natural order may be reversed by $\frac{1}{2}cr-1$ interchanges, when c is even; and by $\frac{1}{2}(c-1)r$ interchanges, when c is odd; *i. e.* the condition for the possibility of forming the reversed natural arrangement from any given arrangement will be the same or the opposite as that for the natural order, according as the number last obtained [$\frac{1}{2}cr-1$, if c is even; and $\frac{1}{2}(c-1)r$, if c is odd] is even or odd. Thus if $r=4$, $c=4$; $\frac{1}{2}cr-1=7$ and the reversed arrangement can always be formed when the natural arrangement cannot, and only then. If $r=5$, $c=5$; $\frac{1}{2}(c-1)r=10$, and the reversed arrangement can be formed when the natural order can be, and only then. *I. e.* with the ordinary "15" puzzle, it is always possible to arrange the numbers in the natural order with the 1 in the right-hand upper square, when it is not possible to do it with the 1 in the left-hand upper square (as Mr. Johnson has remarked in the *postscript* to his note); but on a square board with five squares on a side, a solution is possible from each of the four corners, or not at all.

There are other arrangements beside the natural and reversed natural which may be regarded as solutions of the puzzle, viz: beginning at either corner of the board, the counters may be arranged in their natural order by rows or by columns, there being in all eight such solutions.

The "15" puzzle for the last few weeks has been prominently before the American public, and may safely be said to have engaged the attention of nine out of ten persons of both sexes and of all ages and conditions of the community. But this would not have weighed with the editors to induce them to insert articles upon such a subject in the American Journal of Mathematics, but for the fact that the principle of the game has its root in what all mathematicians of the present day are aware constitutes the most subtle and characteristic conception of modern algebra, viz: the law of dichotomy applicable to the separation of the terms of every complete system of permutations into two natural and indefeasible groups, a law of the inner world of thought, which may be said to prefigure the polar relation of left and right-handed screws, or of objects in space and their reflexions in a mirror. Accordingly the editors have thought that they would be doing no disservice to their science, but rather promoting its interests by exhibiting this *à priori* polar law under a concrete form, through the medium of a game which has taken so strong a hold upon the thought of the country that it may almost be said to have risen to the importance of a national institution. Whoever has made himself master of it may fairly be said to have taken his first lesson in the theory of determinants.

It may be mentioned as a parallel case that Sir William Rowan Hamilton invented, and Jacques & Co., the purveyors of toys and conjuring tricks, in London (from whom it may possibly still be procured), sold a game called the "Eikosion" game, for illustrating certain consequences of the method of quaternions.—EDS.

ERRATA IN No. 2.

Page 118, line 7, for $n] + m$, read $m] + n$.
" 115, " 4, " hx^3, read h^3x^3.
" 116, equation 81, insert (after $=$.
" 118, line 5 from bottom, for c_{xy}, read c^{xy}.
" 118, " 2 " " " $)c^x)$, " $(c\ )$.
" 119, " 6, for c_x, read c^x.
" 123, " 17, " $y = k$, read $dy = k$.
" 130, " 19, " $(-v^n)$, read $(-v)^n$.
" 145, " 4, " $x + \frac{1}{2}$, read $x - \frac{1}{2}$.
" 145, prefix ϕ to equation (314).
" 146, equation (320), insert ϕ after x^{-1}.
" 147, line 25, for ϕ, read x in two places.
" 147, note, line 3, for $2x\partial + \partial$, read $2x\partial + \partial^2$.
" 154, equations (357) and (358), for $-h$, read $+h$.
" 154, " " " " " $[a-1]$, read $[a+1]$.

ERRATA.

Vol. I, p. 383, line 2, *for* is the parameter, *read* is ½ the parameter.

Vol. II, p. 79, line 25, *for* —191, *read* —192.

Vol. II, p. 81, line 18, *for* X, *read* Y.

CLIFFORD TESTIMONIAL FUND.

The friends of Professor Clifford, who was compelled by ill-health to relinquish active work and reside in Madeira, were anxious to present him with a substantial Testimonial in public recognition of his great scientific and literary attainments.

At a meeting held at the Royal Institution, Albemarle Street, on Friday, January 31st, William Spottiswoode, Esq., President of the Royal Society, in the Chair, it was resolved that a Fund should be raised for the above-mentioned purpose, and that the sums received should be placed in the hands of Trustees, for the benefit of Professor Clifford and his family.

In consequence of the lamented death of Professor Clifford (the probability of which at an early period was foreseen) the sums received will be applied to the benefit of his surviving widow and children

Contributions may be paid to the account of "Clifford Testimonial Fund," with Messrs. Robarts, Lubbock & Co., Lombard Street, London.

March, 1879.

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